

Fundamental Groups of a Class of Rational Cuspidal Plane Curves

A. Muhammed Uludağ*

September 15, 2015

Abstract

We compute the presentations of fundamental groups of the complements of a class of rational cuspidal projective plane curves classified by Flenner, Zaidenberg, Fenske and Saito. We use the Zariski-Van Kampen algorithm and exploit the Cremona transformations used in the construction of these curves. We simplify and study these group presentations so obtained and determine if they are abelian, finite or big, i.e. if they contain free non-abelian subgroups. We also study the quotients of these groups to some extent.

1 Introduction

A projective plane curve $C \subset \mathbb{P}^2$ is said to be *cuspidal* if all its singularities are irreducible. It is said to be of *type* (d, m) if C is of degree d and the maximal multiplicity of its singularities is m , i.e. $m = \max_{p \in C}(\text{mult}_p C)$. Rational cuspidal curves of type $(d, d-2)$ and those of type $(d, d-3)$, having at least three cusps has been classified by Flenner and Zaidenberg. It turns out that these curves have exactly three cusps, moreover, the rational 3-cuspidal curves of type $(d, d-2)$ can be obtained from the quadric $\{xy - z^2 = 0\}$, and the rational 3-cuspidal curves of type $(d, d-3)$ can be obtained from the cubic $\{xy^2 - z^3 = 0\}$ by means of Cremona transformations. Our aim in this paper is to compute the fundamental groups of complements of these curves as well as those ones constructed in the ensuing work of Fenske, Sakai and Tono.

Given an algebraic curve $C \subset \mathbb{P}^2$, it is interesting to study the topology of its complement $\mathbb{P}^2 \setminus C$ from the point of view of the classification theory of algebraic curves. The idea is an analogue of the leading principle in the knot theory: in order to understand a knot $\mathcal{K} \subset S^3$, look at the topology of the knot complement $S^3 \setminus \mathcal{K}$. Another strong motivation for studying $\pi_1(\mathbb{P}^2 \setminus C)$ comes from the surface theory: A good knowledge of $\pi_1(\mathbb{P}^2 \setminus C)$ allows one to construct Galois coverings of \mathbb{P}^2 branched at C (see [12] for explicit examples). Moreover,

*Galatasaray University, Department of Mathematics, Çırağan Cad. No. 36, 34357 Beşiktaş İstanbul, Turkey

if $X \rightarrow \mathbb{P}^2$ is a branched covering, with C as the branching locus, one can hope to derive some invariants of X from the invariants of the topology of $\mathbb{P}^2 \setminus C$. The study of $\pi_1(\mathbb{P}^2 \setminus C)$ has been initiated by Zariski in the thirties [13]. Although the prevailing convention in the literature is to call $\pi_1(\mathbb{P}^2 \setminus C)$, by abuse of language, the fundamental group of C , following Degtyarev [2], we shall take the liberty to call it simply *the group of C* . Degtyarev classified the possible groups of curves of degree d with a singular point of multiplicity $d - 2$ and calculated the groups of all quintics (irreducible or not). Hence, groups of curves of degree ≤ 5 are known. This is not the case starting with the sextics. A special class of sextics, namely the rational cuspidal ones, has been classified by Fenske [4], see Theorem 2.7. Their groups are given in Corollary 2.4.

Below we review these classification results followed by Artal's and our results on the fundamental groups. Since group computations uses the explicit construction of these curves, we felt obliged to give a detailed account of classification results.

Acknowledgements. These results are from authors's thesis. Hereby I express my gratitude towards M. Zaidenberg who suggested the problem. I am indebted to Alex Dimca for informing that they are useful and referred, and for his encouragement to publish them. I am thankful to Hakan Ayral for his help with the graphics.

2 Classifications and fundamental groups

We use the following conventions settled in [7]: the multiplicity sequence of a cusp will be called the *type* of this cusp. Recall that if

$$V_{n+1} \rightarrow V_n \rightarrow \cdots \rightarrow V_1 \rightarrow V_0 = \mathbb{C}^2$$

is a minimal resolution of an irreducible analytic curve singularity germ $(C, 0) \subset (\mathbb{C}^2, 0)$, and (C_i, P_i) denotes the proper transform of $(C, 0)$ in V_i , so that $(C_0, P_0) = (C, O)$, then the sequence

$$[m^{(n+1)}, m^{(n)}, \dots, m^{(1)}, m^{(0)}]$$

, where $m^{(i)} := \text{mult}_{P_i} C_i$, is called the *multiplicity sequence* of $(C, 0)$. Evidently, $m^{(i+1)} \leq m^{(i)}$, $m^{(n)} \geq 2$, and $m^{(n+1)} = 1$. The (sub)sequence

$$\underbrace{k, k, k, \dots, k}_{m \text{ times}}$$

will be abbreviated by k_m . For instance, $[kn, k_{n+m}, k - 1]$ is the sequence $[kn, \underbrace{k, k, k, \dots, k}_{n+m \text{ times}}, k - 1]$. Also, the last term of the multiplicity sequence will be omitted. Under this notation, [2] corresponds to a simple cusp and $[2_3] = [2, 2, 2]$ corresponds to a ramphoid cusp.

Theorem 2.1. (Flenner-Zaidenberg [6]) *A rational cuspidal curve of type $(d, d-2)$ with at least three cusps has exactly three cusps. For each $(d, n, m) \in \mathbb{N}^3$ such that $d \geq 4$, $n \geq m > 0$ and $n + m = d - 2$ there is exactly one (up to projective equivalence) such curve C , whose cusps are of types $[d-2]$, $[2_n]$, $[2, 2_m]$. There are no other rational cuspidal curves of type $(d, d-2)$ with number of cusps ≥ 3 .*

Note that an irreducible curve of type $(d, d-1)$ is rational, and has an abelian group by a classical computation due to Zariski.

Theorem 2.2. (Artal [1]) *Let C be as in Theorem 2.1. Then the group of C admits the presentation*

$$\pi_1(\mathbb{P}^2 \setminus C) = \langle a, b \mid (ba)^{d-1} = b^{d-2}, \quad (ba)^k b = a(ba)^k \rangle,$$

where $k \geq 0$ and $2k + 1 = \gcd(2n + 1, 2m + 1)$. Hence, the group depends only on (d, k) . This group is abelian if and only if $k = 0$, finite of order 12 if $(d, n, m) = (4, 1, 1)$, finite of order 840 if $(d, n, m) = (7, 4, 1)$, otherwise it is a big group¹.

Note that the case $(d, n, m) = (4, 1, 1)$ corresponds to the three cuspidal quartic, whose group had been calculated already by Zariski [13].

Taking for example C_1 with $(d, n, m) = (13, 10, 1)$ and C_2 with $(d, n, m) = (13, 7, 4)$, one has the following result.

Corollary 2.1. (Artal [1]) *There exist infinitely many pairs (C_1, C_2) of curves with isomorphic (big) groups, but different homeomorphism types of the pairs (\mathbb{P}^2, C_1) , (\mathbb{P}^2, C_2) .*

Theorem 2.3. (Flenner-Zaidenberg [7]) *A rational cuspidal curve of type $(d, d-3)$ with at least three cusps has exactly three cusps. For each $d = 2n + 3$, $n \geq 1$, there is exactly one (up to projective equivalence) such curve C_n , and the cusps of C_n are of types $[2n, 2_n]$, $[3_n]$, $[2]$. There are no other rational cuspidal curves of type $(d, d-2)$ with number of cusps ≥ 3 .*

Theorem 2.4. *Let C_n be as in Theorem 2.3, where $d = 2n + 3$. Then the group of C_n admits the presentation*

$$\pi_1(\mathbb{P}^2 \setminus C_n) = \langle c, b \mid cbc = bcb, \quad b^n c^{n+2} = c^{n+2} b^n, \quad (b^{-n} c b^2)^{n+1} c^{n^2} = 1 \rangle.$$

This group is big when n is odd and ≥ 7 , abelian when $n \in \{0, 1, 2\}$, and finite of order 8640 for $n = 3$, and finite of order 1560 for $n = 5$.

This is proved in 3.3.1 below.

These classification results have been completed (independently) in the works of Fenske and Sakai-Tono:

¹Recall that we call a group *big* if it has a non-abelian free subgroup.

Theorem 2.5. (*Fenske [4], Sakai-Tono [11]*) Let $n, m \in \mathbb{N}$ be such that $n \geq 1$ and $0 \leq m < n$.

(i) A rational cuspidal curve of type $(d, d-2)$ with exactly one cusp exists if and only if $d = 2n+2$. Such a curve is unique up to projective equivalence, and the type of its cusp is $[2n, 2_{2n}]$.

(ii) A rational cuspidal curve of type $(d, d-2)$ with exactly two cusp exists if and only if

(a) $d = 2n+3$, with types of cusps $[2n+1, 2_n], [2_{n+1}]$;

(b) $d = n+2$, with types of cusps $[n], [2_n]$;

(c) $d = 2n+2$, with types of cusps $[2n, 2_{n+m}], [2_{n-m}]$.

Moreover, all these curves are unique up to projective equivalence.

Theorem 2.6. (*Fenske [4]*)

Let $n, m \in \mathbb{N}$ be such that $n \geq 1$ and $0 \leq m < n$.

(i) A rational cuspidal curve of type $(d, d-3)$ with exactly one cusp exists if and only if

(a) $d = 3n+3$, where the type of the cusp is $[3n, 3_{2n}, 2]$;

(b) $d = 5$, where the type of the cusp is $[2_6]$. These curves are unique up to projective equivalence.

(ii) The only existing rational cuspidal curves of type $(d, d-3)$ with exactly two cusps are the following ones:

	degree	types of cusps
1	7	$[4], [3_3]$
2	6	$[3], [3_2, 2]$
3	5	$[2_4], [2_2]$
4	$2n+3$	$[2n, 2_n], [3_n, 2]$
5	$2n+4$	$[2n+1, 2_n], [3_{n+1}]$
6	$2n+3$	$[2n, 2_{n+1}], [3_n]$
7	$3n+3$	$[3n, 3_{2n}], [2]$
8	$3n+4$	$[3n+1, 3_n], [3_{n+1}]$
9	$3n+3$	$[3n, 3_{n+m}, 2], [3_{n-m}]$
10	$3n+3$	$[3n, 3_{n+m}], [3_{n-m}, 2]$
11	$3n+3$	$[3n+2, 3_n, 2], [3_{n+1}, 2]$

These curves are unique up to projective equivalence.

Considering the cases $d = 6$ in the above theorems leads to a complete classification of rational cuspidal curves of degree 6, given in [4].

Theorem 2.7. (*Fenske [4]*) Up to projective equivalence, rational cuspidal curves of degree 6 are the following ones:

	types of cusps		types of cusps
1	[5]	7	[4, 2 ₃], [2]
2	[4, 2 ₄]	8	[4, 2 ₂], [2 ₂]
3	[3 ₃ , 2]	9	[4], [2 ₄]
4	[3 ₃], [2]	10	[4], [2 ₃], [2]
5	[3 ₂ , 2], [3]	11	[4], [2 ₂], [2 ₂]
6	[3 ₂], [3, 2]		

In fact, in [4] the following longer list of curves is constructed.

Theorem 2.8. (Fenske [4]) *Let $d \geq 2$ and $0 \leq m < n$ be integers. The following rational cuspidal plane curves exist:*

	type of curves (d, m)	type of cusps
1	$(kn + k, kn)$	$[kn, k_{n+m}, k - 1], [k_{n-m}]$
1a	$(kn + k, kn)$	$[kn, k_{2n}, k - 1]$
2	$(kn + k, kn)$	$[kn, k_{n+m}], [k_{n-m}, k - 1]$
2a	$(kn + k, kn)$	$[kn, k_{2n}], [k - 1]$
3	$(kn + k + 1, kn + 1)$	$[kn + 1, k_n], [k_{n+1}]$
4	$(kn + k + 1, kn)$	$[kn, k_{n+1}], [(k + 1)_n]$
5	$(kn + k + 1, kn)$	$[kn, k_n], [(k + 1)_n, k]$
6	$(kn + k + 2, kn + 1)$	$[kn + 1, k_n], [(k + 1)_{n+1}]$
7	$(kn + 2k - 1, kn + k - 1)$	$[kn + k - 1, k_n, k - 1], [k_{n+1}, k - 1]$
8	$(n + 2, n)$	$[n], [2_n]$

Beware of the following exception in the table above: In case $n = 1$, the curves (4) and (5) are of type $(kn + k + 1, k + 1)$, instead of $(kn + k + 1, nk)$.

Theorem 2.9. *Groups of the curves in Theorem 2.8 are as follows:*

1	$\langle \alpha, \beta, y \mid \alpha = y^{-k} \beta^{k-1}, [\beta, y^k] = \alpha(\alpha\beta)^m = \beta(\alpha\beta)^{n-m} = 1 \rangle$
1a	abelian
2	$\langle \alpha, \beta, y \mid \alpha = y^{-k} \beta^{k-1}, [\beta, y^k] = \alpha(\alpha\beta)^{n-m} = \beta(\alpha\beta)^m = 1 \rangle$
2a	$\langle \beta, y \mid y^k = \beta^{k-1}, \beta^{n+1} = 1 \rangle$
3	abelian
4	$\langle x, y \mid (xy)^{n+1} y^{n^2} = [y, (xy)^k] = 1, x^k y^n x = (xy)^k \rangle$
5	abelian
6	abelian
7	abelian
8	abelian

Groups (1) are central extensions of the group $\mathbb{Z}_k * \mathbb{Z}_j$, where $j := \text{g.c.d.}(mk + k - 1, n + 1)$. Thus, they are abelian if $j = 1$, and big if $j \geq 2$. Groups (2) are central extensions of the group $\mathbb{Z}_k * \mathbb{Z}_j$, where $j := \text{g.c.d.}(1 + mk, n + 1)$. Thus, they are abelian if $j = 1$, and big if $j \geq 2$. The same conclusion is true for the groups (2a), where this time $j := \text{g.c.d.}(k - 1, n + 1)$. Groups (4) are

abelian if $j := (n+1, k) = 1$ or $n = 1$. Otherwise, they are big with the following exceptions:

- (i) If $(n, k) = (3, 2)$, then the group is finite non-abelian of order 72 (the degree of the curve is 9).
- (ii) If $(n, k) = (5, 2)$, then the group is finite non-abelian of order 1560 (the degree of the curve is 13)
- (iii) If $(n, k) = (2, 3)$, then the group is finite non-abelian of order 240 (the degree of the curve is 10).

This is proved in 3.3.2-3.3.8 below.

Corollary 2.2. (i) The group of a rational unicuspidal curve of type $(d, d-2)$ is abelian,

(ii) The group of a rational two-cuspidal curve C of type $(d, d-2)$ is abelian unless C is one of the curves described in Theorem 2.5-(ii)c, and $j := \text{g.c.d.}(2m+1, n+1) \neq 1$. In this case, the group of C is the big group with the following presentation

$$\langle y, \beta \mid [\beta, y^2] = y^{-2m-2}\beta^{2m+1} = y^{2m-2n}\beta^{2n-2m+1} = 1 \rangle.$$

This group is a central extension of $\mathbb{Z}_2 * \mathbb{Z}_j$.

Proof. (i) This is the case (1a) with $k = 2$ in Theorem 2.9.

(ii) (a) This is the case (3) with $k = 2$ in Theorem 2.9.

(b) This is the case (8) in Theorem 2.9.

(c) This is the case (1) with $k = 2$ in Theorem 2.9. One obtains the presentation easily by substituting $\alpha = y^{-3}\beta$.

Corollary 2.3. (i) The group of a rational unicuspidal curve of type $(d, d-3)$ is abelian.

(ii) Groups of rational two-cuspidal curves of type $(d, d-3)$ are given below:

	degree	group
1	7	abelian
2	6	$\mathbb{Z}_2 * \mathbb{Z}_3$
3	5	abelian
4	$2n+3$	abelian
5	$2n+4$	abelian
6	$2n+3$	$\langle x, y \mid xy^n x = yxy, (xy)^{n+1}y^{n^2} = [y, xyx] = 1 \rangle$
7	$3n+3$	$\langle y, \beta \mid \beta^2 = y^3, \beta^{n+1} = 1 \rangle$
8	$3n+4$	abelian
9	$3n+3$	$\langle \alpha, \beta, y \mid \alpha = y^{-3}\beta^2, [\beta, y^3] = \alpha(\alpha\beta)^m = \beta(\alpha\beta)^{n-m} = 1 \rangle$
10	$3n+3$	$\langle \alpha, \beta, y \mid \alpha = y^{-3}\beta^2, [\beta, y^3] = \alpha(\alpha\beta)^{n-m} = \beta(\alpha\beta)^m = 1 \rangle$
11	$3n+5$	abelian

Groups (6) are abelian if n is even or $n = 1$. Otherwise they are big unless $n = 3$ or $n = 5$, in these cases the group is finite of order 72 and 1560 respectively. Groups (7) are abelian if n is even, and big otherwise. Groups (9) are abelian

if $\text{g.c.d.}(3m+2, n+1) = 1$, and big otherwise. Groups (10) are abelian if $\text{g.c.d.}(3m+1, n+1) = 1$, and big otherwise.

Proof. 1. This is the case (4) in Theorem 2.9 with $k = 3$ and $n = 1$. Hence, the group has the presentation

$$\langle x, y \mid x^3 y x = (xy)^3, \quad (xy)^2 y = [y, (xy)^3] = 1 \rangle,$$

which is easily seen to be abelian, by substituting $(xy)^2 = y^{-1}$ in the commutation relation.

2. This is the case (1) in Theorem 2.9 with $k = 3$, $n = 1$, and $m = 0$. The group is

$$\langle \alpha\beta, y \mid \alpha = y^{-3}\beta^2, \quad [\beta, y^3] = \alpha = \beta\alpha\beta = 1 \rangle.$$

Thus, $y^3 = \beta^2 = 1$, i.e. the group is $\mathbb{Z}_2 * \mathbb{Z}_3$.

3. The group of this curve was found to be abelian by Degtyarev [2].

4. This is the case (5) with $k = 2$ in Theorem 2.9. The group is thus abelian.

5. This is the case (6) with $k = 2$ in Theorem 2.9, and the group is abelian.

6. This is the case (4) with $k = 2$ in Theorem 2.9.

7. This is the case (2a) with $k = 3$ in Theorem 2.9.

8. This is the case (3) with $k = 3$ in Theorem 2.9.

9. This is the case (1) with $k = 3$ in Theorem 2.9.

10. This is the case (2) with $k = 3$ in Theorem 2.9.

11. This is the case (1) with $k = 3$ in Theorem 2.9.

Corollary 2.4. *Groups of rational cuspidal sextics are listed below.*

	types of cusps	group
1	[5]	abelian
2	[4, 2 ₄]	abelian
3	[3 ₃ , 2]	abelian
4	[3 ₃], [2]	$\mathbb{Z}_2 * \mathbb{Z}_3$
5	[3 ₂ , 2], [3]	$\mathbb{Z}_2 * \mathbb{Z}_3$
6	[3 ₂], [3, 2]	$\mathbb{Z}_2 * \mathbb{Z}_3$
7	[4, 2 ₃], [2]	$\mathbb{Z}_2 * \mathbb{Z}_3$
8	[4, 2 ₂], [2 ₂]	abelian
9	[4], [2 ₄]	abelian
10	[4], [2 ₃], [2]	abelian
11	[4], [2 ₂], [2 ₂]	$\langle a, b \mid (ab)^5 = b^4, \quad a(ba)^2 = (ba)^2b \rangle$

Proof. 1-2-3: These sextics are unicuspidal, hence their groups are abelian by Corollaries 2.2 and 2.3.

4. This is the case (7) in Corollary 2.3 with $n = 1$.

5. This is the case (2) in Corollary 2.3.

6. This is the case (10) in Corollary 2.3 with $n = 1$ and $m = 0$. Thus $j = 1$, and the group is abelian.

7. This is the curve in Corollary 2.2-(c) with $n = 2$ and $m = 1$. Thus $j = 3$, and the group has the presentation

$$\langle y, \beta \mid [\beta, y^2] = y^{-4} \beta^3 = y^{-2} \beta^3 = 1 \rangle,$$

which easily seen to be isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_3$.

8. This is the curve in Corollary 2.2-(c) with $n = 2$ and $m = 0$. Thus $j = 1$, and the group is abelian.

9. This is the curve in Corollary 2.2-(b).

10. This is the curve in Theorem 2.2 with $(d, n, m) = (6, 3, 1)$. Thus, $k = 1$, and the group is abelian.

11. This is the curve in Theorem 2.2 with $(d, n, m) = (6, 3, 1)$, and $k = 2$.

T. Fenske began the classification of rational cuspidal curves of type $(d, d-4)$. Recall that a curve C is said to be *unobstructed* if $H^2(\Theta_V \langle D \rangle) = 0$, where $(V, D) \rightarrow (\mathbb{P}^2, C)$ is a minimal embedded resolution of singularities of C , and $\theta_V \langle D \rangle$ is the sheaf of holomorphic vector fields on V tangent along D .

Theorem 2.10. (Fenske [5]) *For each $n \geq 1$ there exists a rational cuspidal plane curve of type $(d, d-4)$, where $d = \deg C_n = 3n + 4$. This curve has three cusps of types $[3n, 3_n]$, $[4_n, 2_2]$, $[2_1]$. The curve C_n is rectifiable² and unique up to a projective equivalence. Moreover, any unobstructed rational cuspidal curve of type $(d, d-4)$ is projectively equivalent to a curve of this type.*

Theorem 2.11. *Groups of the curves in Theorem 2.10 admit the presentation*

$$\pi_1(\mathbb{P}^2 \setminus \tilde{C}) = \langle a, \gamma \mid a\gamma a = \gamma a^{n+1} \gamma, \quad [a^n, \gamma^3] = a^{3n^2+2n} (\gamma^3 a)^{n+1} = 1 \rangle.$$

This group is big provided $3 \mid (n+1)$ and $n > 6$.

The proof of this theorem is given in 3.3.9.

3 Calculations

Conventions. Throughout this work, we shall use the following conventions: If $\alpha, \beta : [0, 1] \rightarrow T$ are two paths in a topological space T , then the product $\alpha \cdot \beta$ is defined provided that $\alpha(1) = \beta(0)$, and one has

$$\alpha \cdot \beta(t) := \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2}, \\ \beta(2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

If α is a path in T with $\alpha(0) = \alpha(1) = * \in T$, we shall take the freedom to talk about α as an element of $\pi_1(T, *)$, ignoring the fact that the elements of $\pi_1(T, *)$ are equivalence classes of such paths under the homotopy. Also, when this do not lead to a confusion, we shall write $\pi_1(T)$ instead of $\pi_1(T, *)$, omitting the base point.

²i.e., it is equivalent to a line up to the action of the Cremona group $\mathbf{Bir}(\mathbb{P}^2)$ of birational transformations of the projective plane.

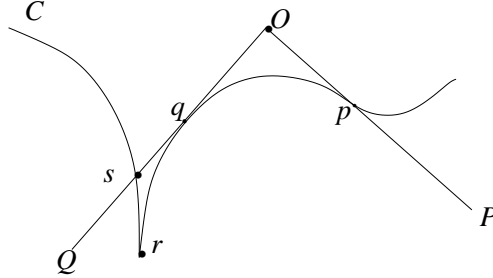


Figure 1

3.1 Groups of the curves in Theorem 2.3

3.1.1 Construction of the curves

Let C be the cubic defined by the equation $x^2z - y^3 = 0$. Then C has a unique, simple cusp at the point $r = [0 : 0 : 1]$, and a unique, simple inflection point at the point $p = [1 : 0 : 0]$. Denote by P the tangent line to C at p . In order to transform C to C_n by means of appropriate Cremona transformations, we begin by taking an arbitrary point $q \in C \setminus \{r, p\}$. Let Q be the tangent to C at q . Then Q intersects C at a second point s , and the lines P and Q intersect at a point $O \notin C$ (Figure 1).

By blowing-up the point O , we obtain a Hirzebruch surface X ; let E be its exceptional section. Let $e := Q \cap E$. We apply an elementary transformation (or Nagata transformation) at the point e followed by an elementary transformation at the point s . Denote by X_1 the Hirzebruch surface so obtained, by Q_1, P_1 the fibers replacing Q, P and by \tilde{C}_1, E_1 the proper transforms of C, E respectively. Then $3p_1 := C_1.P_1$, $2q_1 + s_1 := C_1.Q_1$, $e_1 := E_1.Q_1$. Now we apply an elementary transformation at s_1 followed by an elementary transformation at e_1 . We get another Hirzebruch surface X_2 , with fibers Q_2, P_2 replacing Q_1, P_1 and with proper transforms \tilde{C}_2, E_2 of \tilde{C}_1, E_1 respectively. Iterating this procedure n times gives a Hirzebruch surface X_n with exceptional section E_n satisfying $E_n^2 = -1$. After the contraction of E_n , we turn back to \mathbb{P}^2 . Then $C_n \subset \mathbb{P}^2$ is the image of \tilde{C}_n .

Composition of these birational maps gives a biholomorphism

$$\mathbb{P}^2 \setminus (C \cup P \cup Q) \xrightarrow{\simeq} \mathbb{P}^2 \setminus (C_n \cup P_n \cup Q_n),$$

and hence an isomorphism

$$\pi_1(\mathbb{P}^2 \setminus (C \cup P \cup Q)) \simeq \pi_1(\mathbb{P}^2 \setminus (C_n \cup P_n \cup Q_n)).$$

So, the group of C_n can be deduced from $\pi_1(\mathbb{P}^2 \setminus (C \cup P \cup Q))$ by adding the relations corresponding to the gluing of the lines P_n and Q_n .

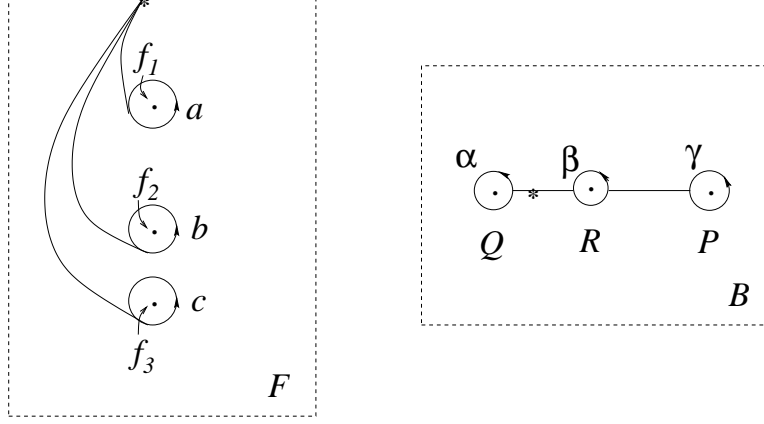


Figure 3

One can identify the fibers with \mathbb{C} e.g. by taking $*$ to be the origin in \mathbb{C}^2 , F to be the y -axis, and B to be the x -axis. Then the projection $\mathbb{C}^2 \rightarrow F$ gives the desired identification.

Choose positively oriented simple loops $a, b, c \in \pi_1(F', *)$ around f_1, f_2, f_3 and the loops $\alpha, \beta, \gamma \in \pi_1(B', *)$ as in Figure (3). Note that $\pi_1(B', *) = \langle \alpha, \beta \rangle$.

The local monodromy of pr' around the points p, q, r and s is well known. The monodromy around q gives the relations

$$\begin{aligned} \alpha^{-1}a\alpha &= b & (\mathcal{R}_1) \\ \alpha^{-1}b\alpha &= bab^{-1} & (\mathcal{R}_2), \end{aligned}$$

and the monodromy around s gives the relation

$$\alpha^{-1}c\alpha = c \Leftrightarrow [\alpha, c] = 1 \quad (\mathcal{R}_3).$$

One has $a = \alpha b \alpha^{-1}$; by (\mathcal{R}_1) , substituting this in the relation (\mathcal{R}_2) we obtain

$$(\alpha b)^2 = (b\alpha)^2 \quad (\mathcal{R}_4).$$

The relation obtained from the monodromy around R gives the cusp relation

$$cbc = bcb \quad (\mathcal{R}_5)$$

(recall that we glue R back to $\mathbb{P}^2 \setminus (C \cup P \cup Q \cup R)$). Since $\pi_1(B') = \langle \alpha, \beta \rangle$, it is not necessary to calculate the relations obtained from the monodromy around P ; these can be derived from the ones we have found. To sum up, we have the presentation

$$\begin{aligned} \pi_1(\mathbb{P}^2 \setminus (C \cup P \cup Q)) = \\ \langle a, b, c, \alpha, \gamma \mid a = \alpha b \alpha^{-1}, (\alpha b)^2 = (b\alpha)^2, cbc = bcb, [\alpha, c] = 1, cba\gamma\alpha = 1 \rangle, \end{aligned}$$

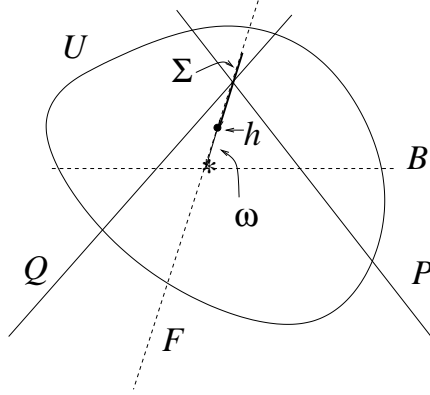


Figure 4

where the last relation $cba\gamma\alpha = 1$ comes from the loop vanishing at infinity. This can be seen as follows: Clearly, cba is a loop in F surrounding the points f_1, f_2, f_3 . Let Σ be a small disc in F containing the point O , and let σ be its boundary. Let \mathbb{R} be the real line in F , put $h := \sigma \cap \mathbb{R}$, and let ω be the real line segment $*h$. Define the positively oriented loop ρ as

$$\rho := \omega \cdot \sigma \cdot \omega^{-1}.$$

Then one has the relation $\rho cba = 1$ since $F = \mathbb{P}^1 = S^2$ is a sphere. Let U be a small neighborhood of O in $\mathbb{P}^2 \setminus (C \cup P \cup Q)$. Then clearly U is biholomorphic to $\Delta^* \times \Delta^*$, where Δ^* is the punctured disc (see Figure (4)). Hence, $\pi_1(U) = \mathbb{Z}^2 = \langle \alpha, \gamma \mid [\alpha, \gamma] = 1 \rangle$, and it is easy to see that ρ is homotopic to $\alpha\gamma$.

Using (\mathcal{R}_1) and $[\alpha, \gamma] = 1$, the relation $cba\gamma\alpha = 1$ becomes

$$cbab\gamma = 1 \quad (\mathcal{R}_6).$$

Eliminating the generators a and γ from this presentation, we get

$$\begin{aligned} \pi_1(\mathbb{P}^2 \setminus (C \cup P \cup Q)) &= \langle b, c, \alpha \mid (\alpha b)^2 = (b\alpha)^2, cbc = bcb, [\alpha, c] = 1 \rangle \\ &= \pi_1(\mathbb{P}^2 \setminus (C_n \cup P_n \cup Q_n)). \end{aligned}$$

To obtain a presentation of the group $= \pi_1(\mathbb{P}^2 \setminus C_n)$, it remains to find the relations corresponding to the gluing of the lines P_n and Q_n . To this end, we introduce the following concept.

Definition 3.1. (*meridian*) Let C be a curve in a surface X , and pick a base point $* \in X \setminus C$. Let Δ be a small analytic disc in X , intersecting C transversally at a unique point q of C . If q is a smooth point of C , a *meridian* of C in X with respect to the base point $*$ is a loop in $X \setminus C$ constructed as follows: Connect $*$ to a point $p \in \partial\Delta$ by means of a path $\omega \subset X \setminus C$ such that $\omega \cap \Delta = p$, and let

$$\mu := \omega^{-1} \cdot \delta \cdot \omega,$$

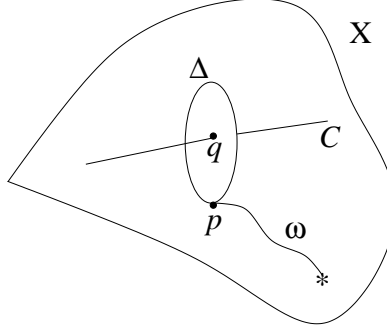


Figure 5

where $\delta := \partial\Delta$, oriented clockwise (Figure (5)). A loop μ_q given by the same construction will be called a *singular meridian at q* if q is a singular point of C .

Lemma 1. *Let $M \triangleleft \pi_1(X \setminus C, *)$ be the subgroup normally generated by the meridians of C . Then*

- (i) $\pi_1(X) = \pi_1(X \setminus C, *)/M$.
- (ii) *If C is irreducible, then any two meridians of C are conjugate in $\pi_1(X \setminus C, *)$. Hence, $M = \langle\langle \mu \rangle\rangle$, where³ μ is any meridian of C .*
- (iii) *Any two singular meridians $\mu_q, \tilde{\mu}_q$ at the singular point $q \in C$ are conjugate.*

Proof. Parts (i)-(ii) are well known [9]. To show (iii), assume that $\mu_q, \tilde{\mu}_q$ are obtained from the discs $\Delta_q, \tilde{\Delta}_q$ intersecting C transversally at q . Let $\sigma : Y \rightarrow X$ be the blow-up of the surface X at the point q , and denote by $Q := \sigma^{-1}(q)$ the exceptional divisor of this blow-up. Then the proper transforms $\sigma^{-1}(\Delta_q), \sigma^{-1}(\tilde{\Delta}_q)$ intersect Q transversally at distinct points of Q , and these points of intersection are smooth in $C \cup Q$. Hence, $\sigma^{-1}(\mu_q), \sigma^{-1}(\tilde{\mu}_q)$ are meridians of Q . Since Q is irreducible, applying the part (ii) to the surface $Y \setminus C$ gives the desired result. \square

For a group G , denote by (g) the conjugacy class of $g \in G$, i.e. $(g) := \{hgh^{-1} : h \in G\}$. Lemma 1 implies that the group of a curve $C \subset \mathbb{P}^2$, supplied with the following data

- (i) Conjugacy classes $(\mu_1), (\mu_2), \dots$ of meridians of C ,
- (ii) Conjugacy classes of singular meridians $(\mu_{q_1}), (\mu_{q_2}), \dots$ of C at singular points q_1, q_2, \dots of C

is a richer invariant of the pair (\mathbb{P}^2, C) than solely the group $\pi_1(\mathbb{P}^2 \setminus C)$.

For the group $\pi_1(\mathbb{P}^2 \setminus (C \cup P \cup Q))$ found above, there are clearly three classes of meridians $(a), (\alpha), (\beta)$, corresponding to the curves C, P, Q , respectively. Consider the loop cb in F , which surrounds the points f_2 and f_3 . Pushing F

³Recall that by $\langle\langle \mu \rangle\rangle$ we denote the normal subgroup generated by μ .

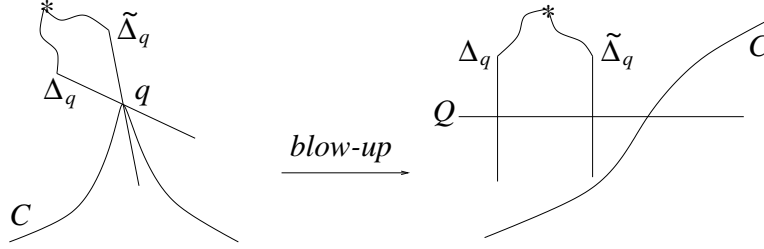


Figure 6

over R , the points f_2, f_3 come together at the cusp r , and the loop cb becomes a loop in R surrounding the cusp r , that is, cb is a singular meridian of $C \cup P \cup Q$ at r , i.e. $(\mu_r) = (cb)$. (The classes $(\mu_O), (\mu_p), (\mu_q), (\mu_s)$ are irrelevant so we will not find them.)

On the other hand, Lemma 1 implies that the relations corresponding to the gluing of the lines Q_n, P_n are of the form $\mu(Q_n) = 1, \mu(P_n) = 1$, where $\mu(Q_n), \mu(P_n)$ are meridians of Q_n and P_n respectively. Finding these meridians will be achieved by Fujita's lemma, which we proceed to explain now.

Let $q \in C$ be an ordinary double point, and take a small neighborhood X' of q such that $X' \cap C$ consists of two branches C_1 and C_2 satisfying $C_1 \cap C_2 = \{q\}$. Pick an intermediate base point $*' \in X' \setminus C$, and take meridians μ'_1 of C_1 in $X' \setminus C_2$ and μ'_2 of C_2 in $X' \setminus C_1$ with respect to $*'$. Let ω be a path in $X \setminus C$ connecting $*$ to $*'$, and define

$$\mu_1 := \omega^{-1} \cdot \mu'_1 \cdot \omega, \quad \mu_2 := \omega^{-1} \cdot \mu'_2 \cdot \omega.$$

Clearly, μ_1, μ_2 are meridians of C in X with respect to $*$ and they commute. Moreover, $\mu_1 \cdot \mu_2$ is homotopic to a singular meridian of C at q . Hence, we have the following lemma:

Lemma 2. (Fujita [8]) *Let $\sigma : X \mapsto Y$ be the blowing up of q , and put $Q := \sigma^{-1}(q)$. Identify $X \setminus \{Q\}$ with $Y \setminus \{q\}$. Then $\mu_1 \cdot \mu_2$ is a meridian of Q in Y with respect to $*$. Moreover, as Q is irreducible, by Lemma 1 one has*

$$\pi_1(X \setminus C, *) = \pi_1(Y \setminus (C \cup Q), *) / \ll \mu_1 \cdot \mu_2 \gg.$$

3.1.3 Meridians of P_n and Q_n

Turning back to our search for the relations introduced in $\pi_1(\mathbb{P}^2 \setminus (C_n \cup P_n \cup Q_n))$ after the gluing of P_n, Q_n , we first note that one can apply Fujita's Lemma to the loops α, γ , which are meridians of P and Q respectively, and obtain a meridian of E . The blowing-up of the point $E \cap P$ will give $\gamma(\alpha\gamma)$ as a meridian of P_1 , by induction we obtain $\mu(P_n) := \gamma(\alpha\gamma)^n$ as a meridian of P_n . Recalling that $[\alpha, \gamma] = 1$, this gives the relation

$$\alpha^n \gamma^{n+1} = 1.$$

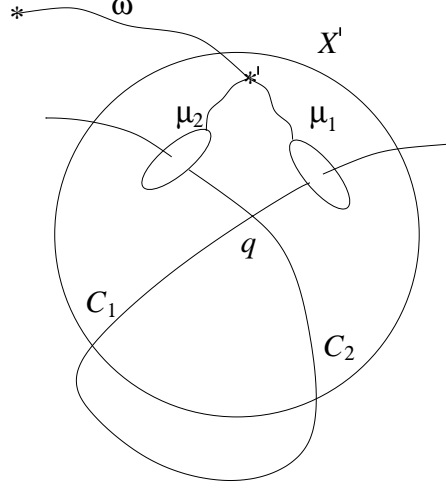


Figure 7

Substituting γ from (\mathcal{R}_6) we get,

$$\alpha^n (cbab)^{-(n+1)} = 1 \quad (\mathcal{R}_7).$$

The construction of C_n also shows that $\mu_p := \alpha^{n-1}\gamma^n$ is a singular meridian of C_n at p_n , since it is a meridian of the exceptional line of the blow-up at p . Setting $\mu(P_n) = 1$ yields that $\mu_p = (\alpha\gamma)^{-1}$ is a singular meridian of C_n at p_n .

To find a meridian of Q_n , first define a meridian $\tilde{\alpha}$ of Q as shown in figure (8). That is, take a small disc Δ intersecting Q transversally above the point s , and take a path ω joining Δ to a neighborhood f_3 , and continuing to $*$ in F' along the loop c . Let $\tilde{\alpha} := \omega \cdot \delta \cdot \omega^{-1}$. Then the blowing up of the point s will give $\tilde{\alpha}c$ as a meridian of Q_1 . A recursive application of Fujita's Lemma gives $\mu(Q_n) := \tilde{\alpha}c^n$ as a meridian of Q_n .

Note that the construction of C_n also shows that $\mu_q := \alpha\gamma$ is a singular meridian of C_n at q_n , since μ_q is a meridian of the exceptional line of the blow-up of q_n .

3.1.4 Finding $\tilde{\alpha}$

The final step in determining the fundamental group of C_n is to express $\tilde{\alpha}$ in terms of the above presentation of $\pi_1(\mathbb{P}^2 \setminus (C \cup P \cup Q))$. Let us show that $\tilde{\alpha}$ is in fact homotopic to α .

First, let $\tilde{\Delta} := pr^{-1}(\Delta) \cap B$. Then α is clearly homotopic to a loop obtained by connecting (properly) $\partial\tilde{\Delta}$ to the base point $*$. $\partial\tilde{\Delta}$ intersect the real axis of B at two points, let $*'$ be the one on the right, which will be used as a temporary base point. Now push the fiber F over $*'$ along the real axis of B . As all the

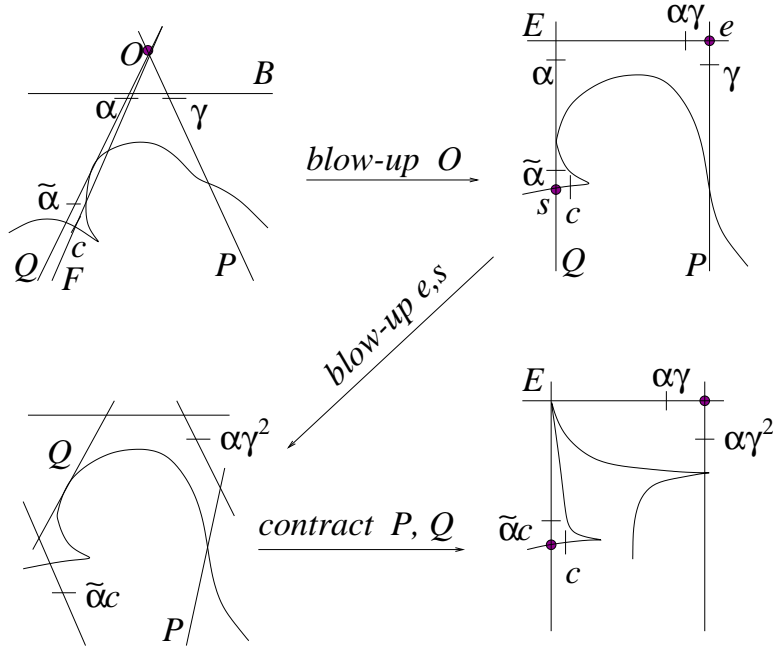


Figure 8

intersection points remains real, it is easy to see that the picture of F stays as in Figure (3).

Next, consider the restriction $pr' : pr^{-1}(\partial\tilde{\Delta}) \rightarrow \partial\tilde{\Delta}$ of pr to the border of the disc $\tilde{\Delta}$. This is a locally trivial fibration, and it can be pictured as in Figure (9), where we have cut $\partial\tilde{\Delta}$ at $*$ ' to give a better picture. Let Σ be a disc in F , containing the upper intersection points f_1 and f_2 of C with F , we suppose that Σ avoids the loop c . Then, as the lower intersection point f_3 is a transversal intersection, we can suppose that the corresponding Leftschetz homeomorphisms $F \rightarrow F_t$, $t \in \partial\tilde{\Delta}$ are constant outside Σ . Hence, the following map gives a homotopy between α and $\tilde{\alpha}$.

$$H(s, t) := \begin{cases} (\alpha(0), \omega(t)), & 0 \leq t \leq s/3 \\ (\alpha\left(\frac{t-s/3}{1-2s/3}\right), \omega(s/3)), & s/3 \leq t \leq 1-s/3 \\ (\alpha(1), \omega(1-t)), & 1-s/3 \leq t \leq 1 \end{cases}$$

Consequently, the vanishing of the meridian of Q_n yields the relation

$$\mu(Q_n) = 1 \Rightarrow \alpha = c^{-n} \quad (\mathcal{R}_8).$$

Substituting α from (\mathcal{R}_8) , (\mathcal{R}_7) becomes

$$c^{n^2}(cbc^{-n}b)^{n+1} = 1.$$

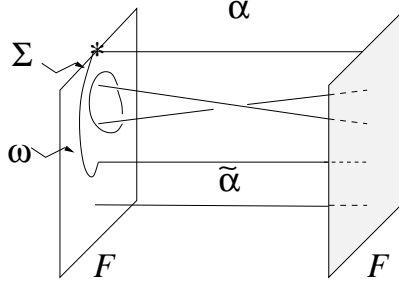


Figure 9

From the cusp relation $cbc = bcb$ it follows that $cbc^k = b^kcb$ for any $k \in \mathbb{Z}$. Using this in the above relation, we get

$$c^{n^2}(b^{-n}cb^2)^{n+1} = 1 \quad (\mathcal{R}_9).$$

To sum up, we have the presentation

$$= \pi_1(\mathbb{P}^2 \setminus C_n) = \langle b, c, \alpha \mid \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_5, \mathcal{R}_7, \mathcal{R}_9 \rangle.$$

3.1.5 Study of the group

Since $\alpha = c^{-n}$ by (\mathcal{R}_7) , the relation (\mathcal{R}_3) is trivialized, and using the cusp relation (\mathcal{R}_4) becomes

$$(bc^{-n})^2 = (c^{-n}b)^2 \Leftrightarrow [b^n, c^{n+2}] = 1.$$

Finally, we have obtained the presentation given in Theorem 2.4.

$$G_n := \pi_1(\mathbb{P}^2 \setminus C_n) = \langle b, c \mid cbc = bcb, [b^n, c^{n+2}] = 1, (b^{-n}cb^2)^{n+1}c^{n^2} = 1 \rangle.$$

Notice that (c) is the unique class of meridian in G_n . Also, the singular meridian $\mu_r = cb$ of C is unchanged during the birational transformations, so that μ_r is a singular meridian of C_n at r_n . Other singular meridians are, as found above, $\mu_q = \alpha\gamma$ and $\mu_p = (\alpha\gamma)^{-1}$. One has $(\alpha\gamma)^{-1} = cba = cbab\alpha^{-1} = cbc^{-n}bc^n$.

It is easily seen that $G_0 = \mathbb{Z}/3\mathbb{Z}$. Let us now show that $G_1 = \mathbb{Z}/5\mathbb{Z}$ and $G_2 = \mathbb{Z}/7\mathbb{Z}$. One has

$$G_1 = \langle c, b, \mid cbc = bcb, [b, c^3] = 1, (b^{-1}cb^2)^2c = 1 \rangle.$$

Expanding the last relation, we get

$$b^{-1}c \cdot bcb^2 \cdot c = b^{-1}c \cdot c^2bc \cdot c = b^{-1}c^3bc^2 = c^5.$$

Thus,

$$[b, c^3] = 1 \Rightarrow [b, c^6] = [b, c^{-1}] = [b, c] = 1 \Rightarrow b = c.$$

Hence, G_1 is generated by b , and $G_1 = \mathbb{Z}/5\mathbb{Z}$. The group G_2 has the presentation

$$G_2 = \langle c, b \mid cbc = bcb, \quad [b^2, c^4] = 1, \quad (b^{-2}cb^2)^3c^4 = 1 \rangle.$$

Again, expanding the last relation we get

$$b^{-2}c^3b^2c^4 = 1 \Rightarrow c^7 = 1.$$

Thus,

$$[b^2, c^4] = 1 \Rightarrow [b^6, c^8] = [b^{-1}, c] = [b, c] = 1 \Rightarrow c = b.$$

Hence, G_2 is generated by b , and one has $G_2 = \mathbb{Z}/7\mathbb{Z}$. The group G_3 is found to be finite of order 8640 by using the programme Maple. The order of the group G_5 is calculated to be 1560 by Artal by the help of the programme GAP.

Definition 3.2. (*residual group*) For an element $a \in G$ of an arbitrary group G , the group $G/\langle\langle a \rangle\rangle$ will be denoted by $G(a)$. If $G_C := \pi_1(\mathbb{P}^2 \setminus C)$ is the group of an irreducible curve C , with μ a meridian of C , then for $k \in \mathbb{N}$, we will call the group $G_C(\mu^k)$ a *residual group* of G_C and denote it by $G_C(k)$. The group $G_C(\mu_p^k)$, where μ_p is a singular meridian of C at a singular point $p \in C$, will be called a *residual group of G_C at p* and denoted by $G_C^p(k)$. Note that the groups $G_C(k)$ do not depend on the particular meridian μ chosen, since by Lemma 1(ii), for an irreducible curve, any two meridians are conjugate. In view of Lemma 1(iii), this is also true for the groups $G_C^p(k)$.

Proposition 3.1. *If n is odd, and $k|n$, then there is a surjection*

$$G_n(k) \twoheadrightarrow T_{2,3,k} = \langle x, y, z \mid x^2 = y^3 = z^k = xyz = 1 \rangle$$

onto the triangle group $T_{2,3,k}$. Hence, G_n is big for odd $n \geq 7$.

Proof. For the last assertion, it is known that the group $T_{2,3,k}$ is big if $k \geq 7$. Putting $n = k$, we get that the groups $G_n(n)$ are big for $n \geq 7$ odd, so that G_n is big if $n \geq 7$ is odd.

Now let us establish the surjection claimed. If $b^k = 1$, then $c^n = b^n = 1$ since $k|n$. This gives the presentation

$$G_n(k) = \langle b, c \mid cbc = bcb, \quad (cb^2)^{n+1} = 1, \quad b^k = 1 \rangle.$$

Following an idea due to Artal [1], we apply the transformation $x = cbc$, $y = cb$ ($\Leftrightarrow c = y^{-1}x$, $b = x^{-1}y^2$) to obtain

$$G_n(k) = \langle x, y \mid x^2 = y^3, \quad (yx^{-1}y^2)^{n+1} = 1, \quad (y^{-1}x)^k = 1 \rangle.$$

Note that $yx^{-1}y^2 = yxy^{-1}$ since $x^2 = y^3$. Thus,

$$G_n(k) = \langle x, y \mid x^2 = y^3, \quad x^{n+1} = 1, \quad (y^{-1}x)^k = 1 \rangle.$$

Let H be the quotient of this group by the relation $x^2 = y^3 = 1$ (note that x^2 is central). Then the relation $x^{n+1} = 1$ is killed if n is odd, and we get the desired result:

$$H = \langle x, y \mid x^2 = y^3 = (y^{-1}x)^k = 1 \rangle = T_{2,3,k}. \quad \square$$

This completes the proof of Theorem 2.4.

3.2 Groups of the curves (1)-(1a) in Theorem 2.9

3.2.1 Construction of the curves

Before passing to the construction of the curves, let us fix some notations following [4].

Notation. Let $\sigma_O : X_1 \rightarrow \mathbb{P}^2$ be the blow-up of \mathbb{P}^2 at the point $O \in \mathbb{P}^2$. Denote by E_1 the exceptional divisor of this blow-up. For any curve $C \subset \mathbb{P}^2$, the proper preimage of C in the Hirzebruch surface X_1 will be denoted by C_1 . Now let X_i be a Hirzebruch surface. Then X_i is a ruled surface, whose horizontal section is denoted by E_i . Let $p \in X_i$, and let L_i be the fiber of the ruling passing through p . The surface obtained from X_i by an elementary transformation at the point p will be denoted by X_{i+1} . Recall that this is a birational mapping which consists of blowing-up $p \in X_i$ followed by the contraction of L_i . The fiber replacing L_i will be denoted by L_{i+1} , and for any curve $C_i \in X_i$, the proper transform of C_i in X_{i+1} will be denoted by C_{i+1} .

Let $n, m \in \mathbb{N}$ be such that $0 \leq m < n$, and for $k \geq 2$, let C be the curve defined by the equation $F(x, y, z) := xy^{k-1} - z^k = 0$. Then for $k > 2$, the curve C has a unique singularity at $p := [1 : 0 : 0]$ which is a cusp and $q = [0 : 1 : 0]$ is an inflection point of C of order k . If $k = 2$, then p is a smooth point of C . The line $P := \{y = 0\}$ is the tangent to C at p and $Q := \{x = 0\}$ is the tangent at q , these tangents intersect at the point $O := P \cap Q = [0 : 0 : 1] \notin C$. Blowing-up the point O , we get a Hirzebruch surface X_1 . Let E_1 be its horizontal section, and denote by P_1, Q_1 the proper transforms of P and Q respectively. For $i = 1, 2, \dots, m$ we apply m elementary transformations at the points $E_i \cap P_i$, followed by elementary transformations applied at the points $E_i \cap Q_i$ for $i = m + 1, m + 2, \dots, n$, and we arrive at the Hirzebruch surface X_{n+1} with $E_{n+1}^2 = -n - 1$ (see Figure (12)).

Performing elementary transformations at arbitrary points $s_i \in P_i \setminus E_i$ for $i = n + 1, \dots, 2n$ we obtain the Hirzebruch surface X_{2n+1} with $E_{2n+1}^2 = -1$. Hence, one can contract E_{2n+1} and return to the projective plane \mathbb{P}^2 . Let $\tilde{C}, \tilde{P}, \tilde{Q}$ be the images of respectively $C_{2n+1}, P_{2n+1}, Q_{2n+1}$ under the contraction of E_{2n+1} . Then \tilde{C} is a curve of the family (1).

The curves (1a) are obtained in the same way, except that in this case one applies elementary transformations at the points $E_i \cap P_i$ for $i = 1, 2, \dots, n$ followed by elementary transformations at some points $s_i \in E_i \cap Q_i$ for $i = n + 1, n + 2, \dots, 2n$.

These birational morphisms provides a biholomorphism

$$\mathbb{P}^2 \setminus (C \cup P \cup Q) \xrightarrow{\simeq} \mathbb{P}^2 \setminus (\tilde{C} \cup \tilde{P} \cup \tilde{Q}).$$

One has the induced isomorphism

$$\pi_1(\mathbb{P}^2 \setminus (C \cup P \cup Q)) \simeq \pi_1(\mathbb{P}^2 \setminus (\tilde{C} \cup \tilde{P} \cup \tilde{Q})).$$

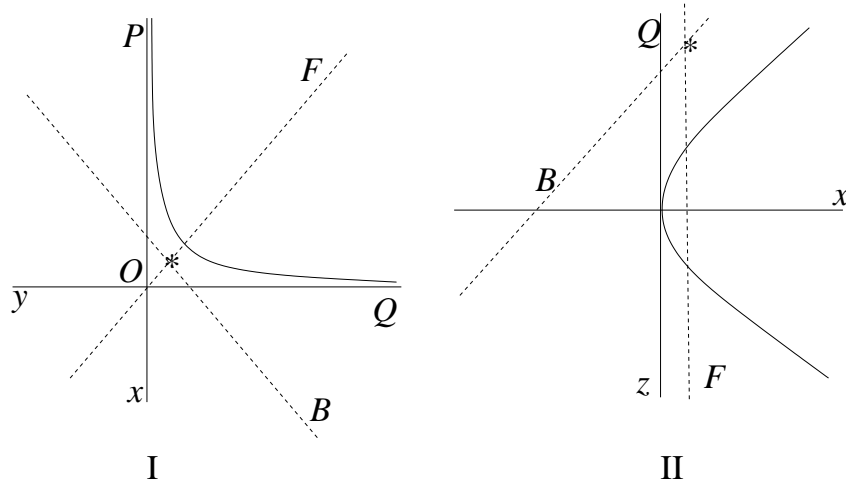


Figure 10

3.2.2 Finding $\pi_1(\mathbb{P}^2 \setminus (C \cup P \cup Q))$

We will apply the Zariski-Van Kampen method to the projection $pr := \mathbb{P}^2 \setminus O \rightarrow \mathbb{P}^1$. Clearly, P, Q are singular fibers of this projection, and it is easy to see that these are the only ones. Indeed, a line passing through $O = [0 : 0 : 1]$ has an equation of the form $ax + by = 0$. Comparing with the equation $xy^{k-1} - z^k = 0$ of C , one obtains $by^k + az^k = 0$, which has multiple solutions if and only if $a = 0, b \neq 0$ or $a \neq 0, b = 0$, corresponding to the lines $P = \{y = 0\}$ and $Q = \{x = 0\}$. Let pr' be the restriction of the projection pr to $\mathbb{P}^2 \setminus (C \cup P \cup Q)$.

Let $L_\infty := \{z = 0\}$, and shift to the affine coordinates in $\mathbb{P}^2 \setminus L_\infty = \mathbb{C}^2$. Let B be the line $\{x + y = \epsilon\}$, where ϵ is a small real number and let $F := \{x = y\}$ be the base fiber (Figure (10)-I). Put $B' := B \setminus (P \cup Q)$ and $F' := F \setminus (\{O\} \cup C)$. If we choose $L_\infty := \{y = 0\}$ and pass to the affine coordinates in $\mathbb{P}^2 \setminus L_\infty = \mathbb{C}^2$, then the real picture of the configuration $C \cup P \cup Q$ will be as it is drawn in Figure (10)-II. Let $* := F \cap B$ be the base point. Identify the fibers with \mathbb{C} via the projection to the z -axis, and take the generators of $\pi_1(B')$ and $\pi_1(F')$ as in Figure (11). The monodromy relations around the singular fiber Q are given by

$$\beta^{-1} a_i \beta = \begin{cases} a_{i+1}, & 1 \leq i \leq k-1, \\ \delta a_1 \delta^{-1}, & i = k. \end{cases}$$

where $\delta := a_k a_{k-1} \dots a_2 a_1$. Setting $a := a_1$, these relations can be expressed as

$$a_i = \beta^{-i+1} a \beta^{i-1}, \quad (\beta a)^k = (a \beta)^k.$$

Hence, one has the presentation

$$\pi_1(\mathbb{P}^2 \setminus (C \cup T \cup L)) = \langle \beta, a \mid (a \beta)^k = (\beta a)^k \rangle.$$

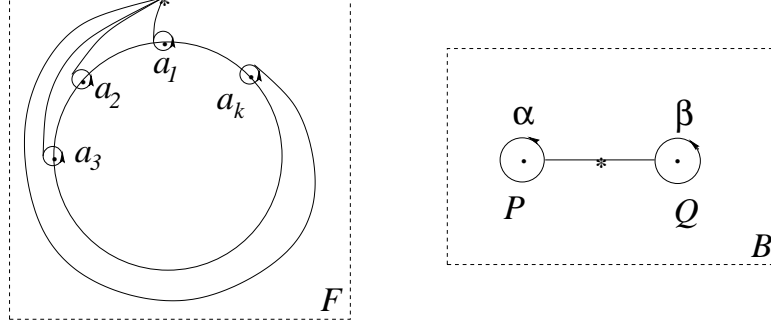


Figure 11

Note that $[\alpha, \beta] = 1$ and $\alpha\beta a_k \cdots a_2 a_1 = 1$. The change of generators $(\beta, a) \Leftrightarrow (\beta, y := \beta a)$ gives a more convenient presentation

$$\pi_1(\mathbb{P}^2 \setminus (C \cup T \cup L)) = \langle \beta, y \mid [\beta, y^n] = 1 \rangle.$$

For the future applications, note that α can be expressed by using the relation

$$\alpha\beta a_k a_{k-1} \cdots a_1 = 1 \Leftrightarrow \alpha = y^{-k} \beta^{k-1}.$$

Note also that $[\alpha, \beta] = 1$. This can be derived either from the above presentation or by applying Fujita's lemma to the meridians α of P and β of Q , with respect to the point $O = P \cup Q$.

3.2.3 Meridians of \tilde{P} and \tilde{Q}

An obvious application of Fujita's lemma yields that $\mu(P_{n+1}) := \alpha(\alpha\beta)^m$ is a meridian of P_{n+1} and $\mu(Q_{n+1}) := \beta(\alpha\beta)^{n-m}$ is a meridian of Q_{n+1} on the surface $X_{n+1} \setminus (C_{n+1} \cup P_{n+1} \cup Q_{n+1} \cup E_{n+1})$. (See Figure (12), where the situation is illustrated for $n = 2, m = 1$, beware that the elementary transformation applied at $e := E_1 \cap P_1$ and the elementary transformation applied at $\tilde{e} := E_2 \cap Q_2$ are shown simultaneously in the figure.) Recall that the subsequent transformations are applied at points $s_i \in E_i \setminus P_i$ for $i = n+1, \dots, 2n$. Since the line Q_{n+1} is not affected by these transformations, $\mu(Q_{n+1})$ is a meridian of \tilde{Q} in $\mathbb{P}^2 \setminus (\tilde{C} \cup \tilde{P} \cup \tilde{Q})$, too.

On the other hand, $\mu(P_{n+1})$ stays to be a meridian of P_{n+i} after an elementary transformation applied at a point $s_i \in E_i \setminus P_i$. This can be seen e.g. by choosing $s_{n+1} = \Delta \cap P_{n+1}$, where Δ is the defining disc of $\mu(P_{n+1})$. Hence, $\mu(P_{n+1})$ is a meridian of \tilde{P} , too.

Denote by \tilde{p} and \tilde{q} the cusps of \tilde{C} . Then, by the construction of the curve, $\alpha\beta$ is a singular meridian at \tilde{p} , and $\beta(\alpha\beta)^{n-m-1}$ is a singular meridian at \tilde{q} .

Setting $\mu(P_{n+1}) = \mu(Q_{n+1}) = 1$, we obtain the presentation

$$G_{(1)} := \pi_1(\mathbb{P}^2 \setminus \tilde{C}) =$$

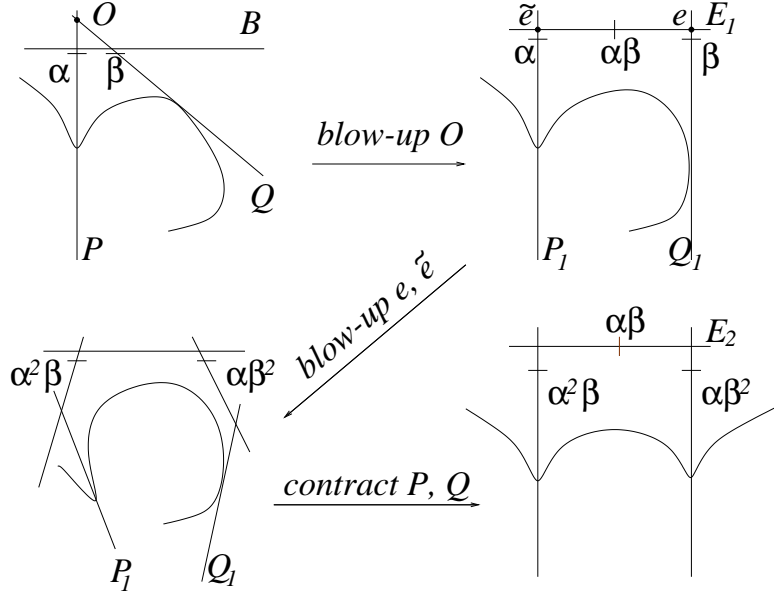


Figure 12

$$\langle \alpha, \beta, y \mid \alpha = y^{-k} \beta^{k-1}, \quad [\beta, y^k] = \alpha(\alpha\beta)^m = \beta(\alpha\beta)^{n-m} = 1 \rangle.$$

A meridian of \tilde{C} can be given as $a = \beta^{-1}y$. After the obvious simplifications, one finds that $\alpha\beta = y^{-k}\beta^k$ is a singular meridian at \tilde{p} , and $(\alpha\beta)^{-1} = y^k\beta^{-k}$ is a singular meridian at \tilde{q} .

It is easy to see that the element y^k is central in the group $G_{(1)}$. Eliminating α in $G_{(1)}(y^k)$, one obtains

$$G_{(1)}(y^k) = \langle \beta, y \mid y^k = \beta^j = 1 \rangle = \mathbb{Z}_k * \mathbb{Z}_j,$$

where $j := \text{g.c.d.}(mk + k - 1, nk - mk + 1) = \text{g.c.d.}(mk + k - 1, kn + k) = \text{g.c.d.}(mk + k - 1, n + 1)$. Hence, this group is big if $j \geq 2$. Finally, the group $G_{(1)}$ is abelian when $j = 1$ by the following trivial lemma:

Lemma 3. *Let G be a group, and $z \in G$ be a central element. If $G(z)$ is cyclic, then G is abelian.*

As for the curves from the family (1a), the same procedure applies. The meridian β of Q stays to be a meridian of \tilde{Q} , so that one has the relation $\beta = 1$, which implies that the fundamental group is generated by just one element $y(=a)$ and thus it is abelian.

Remark. In [3], the authors provide a long argument due to V. Lin, showing the bigness of the group given by the presentation $\langle a, b \mid (ab)^2 = (ba)^2 \rangle$. Here is a simpler proof of a more general assertion:

Proposition 3.2. *Let $n, m \in \mathbb{Z}$ such that $k := \gcd(n, m)$ satisfy $|k| \geq 2$. Then the group*

$$\langle a, b \mid (ab)^n = (ba)^m \rangle$$

is big.

Proof. Put, as above, $x := ab$ and $y := b$. Then the above presentation is written in terms of x, y as

$$\langle x, y \mid x^n = yx^my^{-1} \rangle.$$

Passing to the quotient by the relation $x^k = 1$, we get the group $\mathbb{Z}_k * \mathbb{Z}$, which is big. This can be seen as follows: Let $r \geq 3$ be an integer such that $\gcd(r, k) = 1$. Passing once more to the quotient by the relation $y^r = 1$ gives the group $\mathbb{Z}_k * \mathbb{Z}_r$, and it is well known that the commutator subgroup of this group is the free group of rank $(k-1)(r-1)$. \square

Note that the group $\mathbb{Z}_2 * \mathbb{Z}_2$ is isomorphic to the infinite dihedral group \mathbb{D}_∞ , whose commutator subgroup is \mathbb{Z} , hence this group is solvable and is not big. Also, it can be shown that the commutator subgroup of the group $\langle a, b \mid ab = (ba)^2 \rangle$ is abelian, but not finitely generated.

3.2.4 Groups of the curves (2)-(2a)

These curves are constructed as the curves (1)-(1a) with the following difference: One performs elementary transformations at the points $E_i \cap Q_i$ for $i = 1, 2, \dots, m$ followed by elementary transformations at the points $E_i \cap P_i$ for $i = m+1, m+2, \dots, n$. Finally, for $i = n, n+1, \dots, 2n$ one applies elementary transformations at some points $s_i \in Q_i \setminus E_i$. The curves (2a) are obtained by setting $n = m$ in the above procedure.

The same reasoning as in the case of the curves (1)-(1a) shows that $\mu(Q_{m+1}) := \beta(\alpha\beta)^m$ is a meridian of \tilde{Q} , and $\mu(P_{m+1}) := \alpha(\alpha\beta)^{m-n}$ is a meridian of \tilde{P} . Setting $\mu(Q_{m+1}) = \mu(P_{m+1}) = 1$, we obtain the presentation

$$G_{(2)} := \pi_1(\mathbb{P}^2 \setminus \tilde{C}) =$$

$$\langle \alpha, \beta, y \mid \alpha = y^{-k}\beta^{k-1}, \quad [\beta, y^k] = \beta(\alpha\beta)^m = \alpha(\alpha\beta)^{n-m} = 1 \rangle.$$

Obviously, $\alpha\beta$ and $(\alpha\beta)^{-1}$ are singular meridians at \tilde{q} and \tilde{p} , respectively. A meridian of \tilde{C} can be given as $a = \beta^{-1}y$.

For the curves (2a) we obtain,

$$G_{(2a)} := \pi_1(\mathbb{P}^2 \setminus \tilde{C}) =$$

$$\begin{aligned} \langle \alpha, \beta, y \mid \alpha = y^{-k}\beta^{k-1}, \quad [\beta, y^k] = \alpha^m\beta^{m+1} = \alpha = 1 \rangle. \\ = \langle \beta, y \mid y^{-k} = \beta^{k-1}, \quad \beta^{n+1} = 1 \rangle. \end{aligned}$$

Meridians in this case can be obtained from those of the case (2) by putting $\alpha = 1$.

Again, the element y^k is central in the group G_2 , and one has

$$G_{(2)}(y^k) = \langle y, \beta \mid y^k = 1, \quad \beta^j = 1 \rangle = \mathbb{Z}_k * \mathbb{Z}_j,$$

where this time $j := \text{g.c.d.}(1 + mk, nk - mk + k - 1) = \text{g.c.d.}(1 + mk, nk + k) = \text{g.c.d.}(1 + mk, n + 1)$.

As for the curves (2a) we obtain,

$$G_{(2a)}(y^k) = \langle y, \beta \mid y^k = 1, \quad \beta^j = 1 \rangle = \mathbb{Z}_k * \mathbb{Z}_j,$$

where $j = \text{g.c.d.}(k - 1, n + 1)$.

3.3 Groups of the curves (3)

Let C be the curve defined by the equation $xy^k - z^{k+1} = 0$, the point p be its cusp, q its inflection point, and R be the line $\overline{pq} = \{z = 0\}$. Let, as in the case (1), P, Q be the tangent lines at p, q , respectively.

Blowing-up the inflection point $q = [0 : 1 : 0]$ we get the Hirzebruch surface X_1 . For $i = 1, 2, \dots, n$, we perform elementary transformations at the points $E_i \cap R_i$, followed by elementary transformations at some points $s_i \in Q_i \setminus E_i$, and we end up with a Hirzebruch surface X_{2n+1} with $E_{2n+1}^2 = 1$. Let $\tilde{C}, \tilde{R}, \tilde{Q}$ be the images of $C_{2n+1}, R_{2n+1}, Q_{2n+1}$ in \mathbb{P}^2 under the contraction of E_{2n+1} . Then \tilde{C} is a curve from the family (3). One has the biholomorphic map

$$\mathbb{P}^2 \setminus (C \cup R \cup Q) \xrightarrow{\sim} \mathbb{P}^2 \setminus (\tilde{C} \cup \tilde{R} \cup \tilde{Q}),$$

inducing an isomorphism

$$\pi_1(\mathbb{P}^2 \setminus (C \cup R \cup Q)) \simeq \pi_1(\mathbb{P}^2 \setminus (\tilde{C} \cup \tilde{R} \cup \tilde{Q})).$$

In order to find $\pi_1(\mathbb{P}^2 \setminus (C \cup R \cup Q))$, we shall use the projection from the point $O := P \cap Q$, as before. As the only points of intersection $R \cap C$ are p and q , this projection has only two singular fibers, namely P and Q . Choose the base B and the generic fiber F as in Figure (10). Let $F' := F \setminus (C \cup \{O\} \cup R)$, and choose the generators $a_1, \dots, a_{k+1}, \gamma$ for $\pi_1(F')$ as in Figure (13). The monodromy relations around Q are given by

$$\begin{aligned} \beta^{-1} a_i \beta &= \begin{cases} a_{i+1}, & 1 \leq i \leq k, \\ (\delta \gamma) a_1 (\delta \gamma)^{-1}, & i = k + 1, \end{cases} \\ \beta^{-1} \gamma \beta &= a_1 \gamma a_1^{-1}. \end{aligned}$$

where $\delta := a_{k+1} a_k \dots a_2 a_1$. Observe that, as β is a meridian of Q , and as the elementary transformations are applied at points $s_i \in Q_i \setminus E_i$, β is a meridian of Q_i , and thus it is a meridian of \tilde{Q} . Imposing the relation $\beta = 1$ in the relations found above, we find that $a_1 = a_2 = \dots = a_{k+1}$, and $[a_1, \gamma] = 1$. Since the group $\pi_1(\mathbb{P}^2 \setminus \tilde{C})$ is generated by these elements we conclude that it is abelian.

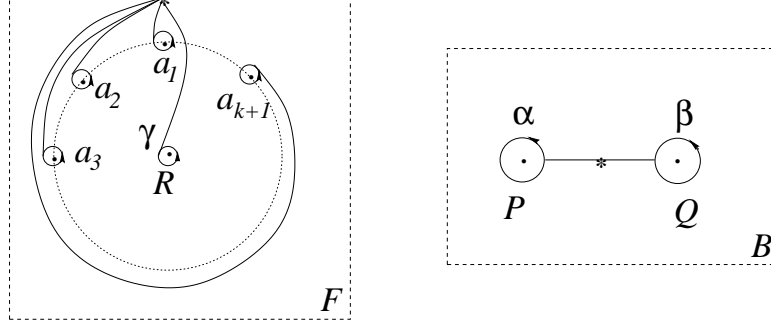


Figure 13

3.4 Groups of the curves (4)

We begin by the curve $C := xy^k - z^{k+1}$. Let $Q := \{x = 0\}$ be the tangent to C at its flex $q := [0 : 1 : 0]$, and let P be a line intersecting C transversally at its cusp $p := [1 : 0 : 0]$, and such that $q \notin P$. Then by Bezout's theorem, P intersects C at one further point $r \neq q$. Let $O := P \cap Q$. Blowing-up O , we get the Hirzebruch surface X_1 , with the horizontal section E_1 with $E_1^2 = -1$. For $i = 1, 2, \dots, n$, apply elementary transformations at the points r_i followed by elementary transformations applied at the points $E_i \cap Q_i$. Then $E_{2n+1}^2 = -1$; contracting it, we turn back to the projective plane \mathbb{P}^2 . Let \tilde{C} , \tilde{P} , \tilde{Q} be the images of C_{2n+1} , P_{2n+1} , Q_{2n+1} under this contraction. Then \tilde{C} is a curve of the family (4), and one has the biholomorphism

$$\mathbb{P}^2 \setminus (C \cup P \cup Q) \xrightarrow{\cong} \mathbb{P}^2 \setminus (\tilde{C} \cup \tilde{P} \cup \tilde{Q}),$$

inducing an isomorphism of the fundamental groups.

To find the group of $C \cup P \cup Q$, we shall use the projection from the point O . In addition to P and Q , this projection has a third singular fiber R , which is a simple tangent to C at a unique, smooth point of C . That P , Q , R are the only singular fibers can be seen by looking at the dual picture. Indeed, by the class formula, one has

$$d^* = 2(g - 1 + k + 1) - (k - 1) = k + 1$$

where $g = 0$ is the genus of C , and d^* is the degree of the dual curve C^* . Now, the point $Q^* \in \mathbb{P}^{2*}$ is a cusp of C^* with multiplicity k . Hence, the line O^* which passes through Q^* should intersect C^* transversally at a unique further point R^* , which is the dual of the simple tangent line R from O to C .

Now we apply the change of coordinates

$$[x : y : z] \Rightarrow [x + y : y : z].$$

In the new coordinates, the equation of C reads as $(x - y)y^k - z^{k+1} = 0$. Let L_∞ be the line $x = 0$, and pass to the affine coordinates $(y/x, z/x)$ in

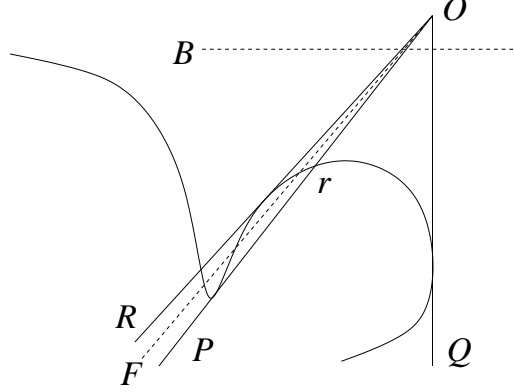


Figure 14

$\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$. Recall that we have the freedom to choose O (or, equivalently, P). So let $O = (1, z_0)$, where z_0 is a big real number. The real picture of the configuration $C \cup P \cup Q \cup R$ is shown in Figure (14). Choose the base fiber F , and the base of the projection as in Figure (14). Put $F' := F \setminus (C \cup O)$ and $B' := B \setminus (P \cup Q \cup R)$. Take the generators b, a_1, a_2, \dots, a_k for $\pi_1(F')$ as in Figure (15)-I and the generators α, β, γ for $\pi_1(B')$ as in Figure (15)-III.

The monodromy around R yields (after setting $\gamma = 1$),

$$a_1 = b \quad (\mathcal{R}_1),$$

and the monodromy around P gives

$$\beta^{-1} a_i \beta = \begin{cases} \delta a_{i+1} \delta^{-1} & 1 \leq i < k, \\ \delta^2 a_1 \delta^{-2} & i = k, \end{cases} \quad (\mathcal{R}_2),$$

where $\delta := a_k a_{k-1} \cdots a_1$. Hence, we have the presentation

$$\pi_1(\mathbb{P}^2 \setminus (C \cup P \cup Q \cup R)) = \langle \beta, b, a_1, a_2, \dots, a_k \mid (\mathcal{R}_1), (\mathcal{R}_2) \rangle$$

Note that $\alpha \beta \delta b = 1$, and that $[\alpha, \beta] = 1$.

An obvious application of Fujita's Lemma shows that βb^n is a meridian of \tilde{P} and $\alpha(\alpha\beta)^n$ is a meridian of \tilde{Q} . Therefore,

$$G_{(4)} := \pi_1(\mathbb{P}^2 \setminus \tilde{C}) = \langle \beta, b, a_1, a_2, \dots, a_k \mid \beta b^n = \alpha(\alpha\beta)^n = 1, (\mathcal{R}_1), (\mathcal{R}_2) \rangle.$$

3.4.1 Study of the group

By using (\mathcal{R}_2) , one can express the generators a_1, a_2, \dots, a_k in terms of b as follows

$$a_i = (\beta\delta)^{-i+1} b (\beta\delta)^{i-1} \quad 1 \leq i \leq k.$$

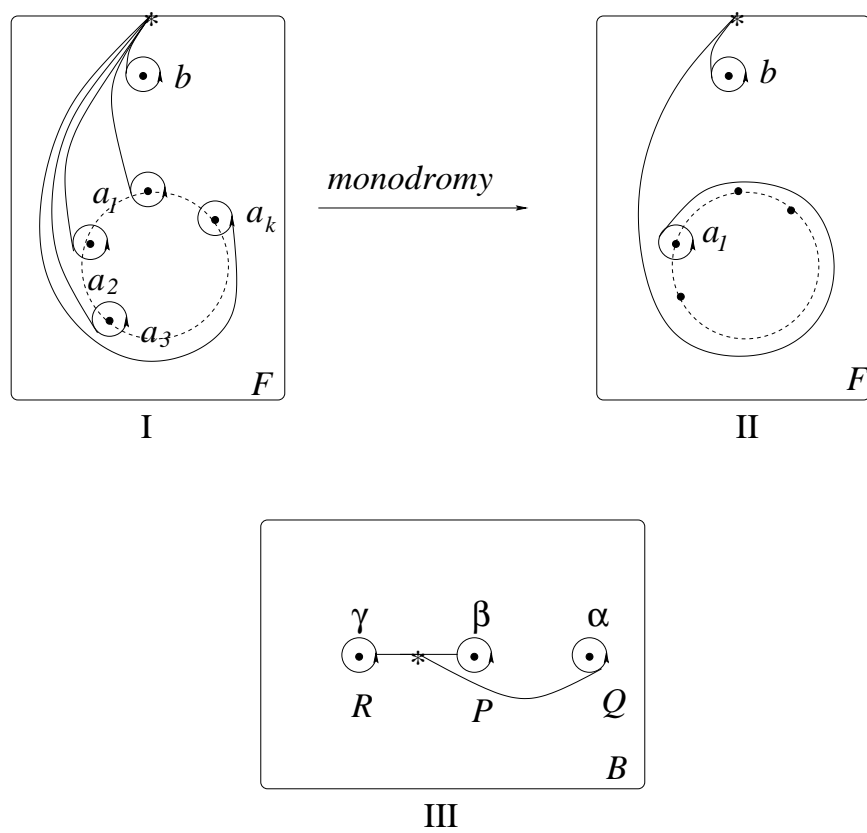


Figure 15

Then, the last relation in (\mathcal{R}_2) reads

$$(\beta\delta)^{-k}b(\beta\delta)^k = \delta b\delta^{-1} \quad (\mathcal{R}_3).$$

For δ , one has

$$\delta = a_k a_{k-1} \cdots a_1 = (\beta\delta)^{-k}(\beta\delta b)^k \quad (\mathcal{R}_4).$$

Since $\alpha = (\beta\delta b)^{-k}$, the relation $\alpha^{n+1}\beta^n = 1$ becomes

$$(\beta\delta b)^{n+1}b^{n^2} = 1 \quad (\mathcal{R}_5),$$

where we have used $\beta = b^{-n}$. This gives the presentation

$$G_{(4)} = \langle \beta, \delta, b \mid \delta = (\beta\delta)^{-k}(\beta\delta b)^k, \quad \beta = b^{-n}, \quad (\beta\delta b)^{n+1}b^{n^2} = 1, \quad (\beta\delta)^{-k}b(\beta\delta)^k = \delta b\delta^{-1} \rangle.$$

Now put $x := \beta\delta$ and $y := b$. Then $\delta = \beta^{-1}x = y^n x$, and one can rewrite the above presentation as

$$G_{(4)} = \langle x, y \mid (xy)^{n+1}y^{n^2} = [y, x^k y^n x] = 1, \quad x^k y^n x = (xy)^k \rangle,$$

since (\mathcal{R}_3) becomes

$$x^{-k}yx^k = y^n xyx^{-1}y^{-n} \Leftrightarrow [y, x^k y^n x] = 1,$$

and for (\mathcal{R}_4) one has

$$y^n x = x^{-k}(xy)^k \Leftrightarrow x^k y^n x = (xy)^k.$$

Finally, (\mathcal{R}_5) is written as $(xy)^{n+1}y^{n^2} = 1$. Note that $y = b$ is a meridian of \tilde{C} .

Obviously, the latter presentation is equivalent to the presentation

$$G_{(4)} = \langle x, y \mid (xy)^{n+1}y^{n^2} = [y, (xy)^k] = 1, \quad x^k y^n x = (xy)^k \rangle \quad (*)$$

To simplify this presentation further, put $z := xy$. Then $x = zy^{-1}$, and one obtains the presentation

$$G_{(4)} = \langle z, y \mid z^{n+1}y^{n^2} = [y, z^k] = 1, \quad (zy^{-1})^k y^n zy^{-1} = z^k \rangle$$

It is readily seen from this presentation that the element z^k is central. Passing to the quotient by z^k gives

$$G_{(4)}(z^k) = \langle z, y \mid z^{n+1}y^{n^2} = 1, \quad (zy^{-1})^{k+1}y^n = z^k = 1 \rangle.$$

Now put $j := \gcd(n+1, k)$. Then one has

$$G_{(4)}(z^k)(y^n) = \langle z, y \mid z^j = y^n = (zy^{-1})^{k+1} = 1 \rangle = T_{j,n,k+1},$$

so that this latter group is big if

$$\frac{1}{j} + \frac{1}{n} + \frac{1}{k+1} < 1. \quad (1)$$

Obviously, $G_{(4)}(z^k)$, and hence also $G_{(4)}$ is abelian if $j = 1$ or $n = 1$. So suppose that $n, j \geq 2$.

First we consider the case $k = 2$. Since $j \geq 2$, this forces $n \geq 3$ to be odd. If $n \geq 7$, then 1 is satisfied. In case $n = 3$ or $n = 5$, 1 is violated.

Now assume $k = 3$. Then $j = 3$ by the assumption $j \geq 2$, which forces $n + 1 \geq 3$ to be a multiple of 3. For $n + 1 \geq 6$, 1 is not violated. For $n = 2$, 1 is violated.

For $k \geq 4$, first assume that k is even. Then the least value that j can take is 2, and the least value that n can take is 3. If $n = 3$, then $j = 4$, and 1 is not violated. But if $n \geq 4$, then 1 is not violated neither.

If $k \geq 4$ is odd, then the least divisor of k is 3, hence $j \geq 3$, and $k \geq 6$. In this case 1 is never violated.

This leaves the cases $(d, n, k) = (9, 3, 2)$, $(d, n, k) = (13, 5, 2)$, and $(n, k) = (10, 2, 3)$ open. Calculations with Maple show that these are finite groups of order 72, 1560 and 240 respectively.

Notice that when $k = 2$, the degree of the curves (4) is $2n + 3$, so it is interesting to compare their groups with the groups in Theorem 3.2.2. For $k = 2$, the relation $x^2 y^n x = (xy)^k$ in the presentation (*) above becomes

$$x^2 y^n x = xyxy \Leftrightarrow xy^n x = yxy \quad (**).$$

Now put

$$\alpha := y, \quad \beta := xy^{n-1}.$$

Then

$$y = \alpha, \quad x = \beta \alpha^{1-n},$$

and the relation (**) becomes the braid relation $\beta \alpha \beta = \alpha \beta \alpha$. On the other hand, the relation $[y, (xy)^k] = 1$ in the presentation (*) is written, in terms of α, β , as $[\alpha, \beta^n] = 1$. Finally, the relation $(xy)^{n+1} y^{n^2} = 1$ in (*) becomes

$$(\beta \alpha^{2-n})^{n+1} \alpha^{n^2} = 1.$$

Hence, the presentation

$$\langle \alpha, \beta \mid \beta \alpha \beta = \alpha \beta \alpha, \quad [\alpha, \beta^n] = (\beta \alpha^{2-n})^{n+1} \alpha^{n^2} = 1 \rangle.$$

3.5 Groups of the curves (5)

Let C be the curve $\{xy^{k-1} - z^k - z^{k-1}y\}$. Its unique singularity is a cusp at the point $p := [1 : 0 : 0]$, and it has a flex of order $k - 1$ at the point $q := [0 : 1 : 0]$. Let P, Q be the tangents to C at p and q . By Bezout's theorem, Q intersect C at a third point r . Blowing-up the point $O := P \cap Q$, we get the Hirzebruch surface X_1 . Perform n elementary transformations at $E_i \cap P_i$, followed by n elementary transformations at the points r_i . One obtains the Hirzebruch surface X_{2n+1} with $E_{2n+1}^2 = -1$. Contraction of E_{2n+1} gives the projective plane \mathbb{P}^2 ; denote by $\tilde{C}, \tilde{P}, \tilde{Q}$ the images of C, P and Q . Then \tilde{C} is a curve of the family (5).

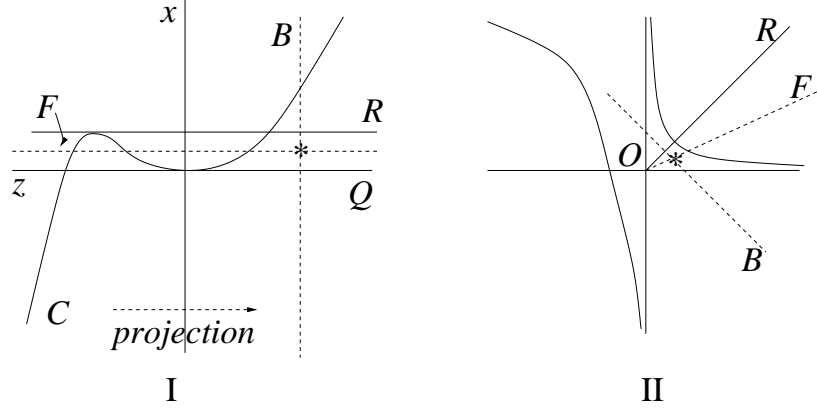


Figure 16

To find $\pi_1(\mathbb{P}^2 \setminus (C \cup P \cup Q))$, we shall use the projection from the point O . Evidently, P and Q are singular fibers of this projection. There is one further singular fiber say R , which is tangent to C at a unique smooth point of C . Indeed, by the class formula, one has $d^* = k + 1$. The point $Q^* \in \mathbb{P}^{2*}$ is a cusp of multiplicity $k - 1$. The line O^* intersect C^* at P^* , which is a smooth point of C^* , and at the cusp Q^* . By Bezout's theorem, O^* should intersect C^* at a third point R^* transversally, which is the point dual to the line R^* . This reasoning shows also that there are no other singular fibers.

In the affine coordinates (x, z) , the equation of C reads $x = z^k + z^{k+1}$, and it is easy to see that the third singular fiber is the tangent line at $z = -k/(k + 1)$.

For k even, the situation is pictured in Figure (16). Choose the base B and the fiber F as in the Figure (16), define F' , B' as usual, and choose the generators for their fundamental groups as in Figure (17).

Set $\delta := a_k a_{k-1} \dots a_2 a_1$. Then the monodromy around Q yields the relations

$$\beta^{-1} a_i \beta = \begin{cases} a_{i+1}, & 1 \leq i \leq k-1, \\ \delta a_1 \delta^{-1}, & i = k, \end{cases} \quad [\beta, b] = 1$$

and the monodromy around R gives

$$a_k = b$$

Now, the relation $\beta^{-1} a_{k-1} \beta = a_k = b$ implies $a_{k-1} = b$, since $[\beta, a_k] = 1$. Similarly, one obtains $a_1 = a_2 = \dots = a_k = b$. This shows that $\pi_1(\mathbb{P}^2 \setminus (C \cup P \cup Q))$ is abelian, which implies that $\pi_1(\mathbb{P}^2 \setminus \tilde{C})$ is abelian, too.

3.6 Groups of the curves (6) in Theorem 2.9

Let C be the curve $\{xy^{k+1} - z^{k+2} - z^{k+1}y = 0\}$, and let Q be its tangent line at the inflection point q . Denote by S the line \overline{pq} . The line Q intersects C

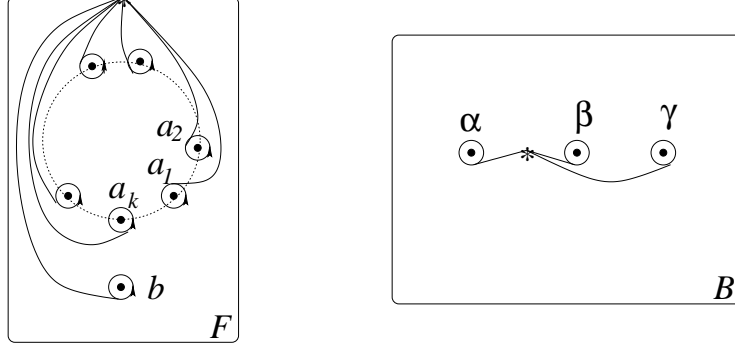


Figure 17

at a second point r . Blowing-up the point q , we get the Hirzebruch surface X_1 with the horizontal section E_1 . For $i = 1, 2, \dots, n$, perform elementary transformations at $E_i \cap S_i$, followed by elementary transformations at the points r_i for $i = n+1, n+2, \dots, 2n$. We end up with the Hirzebruch surface X_{2n+1} , with $E_{2n+1} = -1$. Contraction of E_{2n+1} gives the projective plane \mathbb{P}^2 , denote by $\tilde{S}, \tilde{Q}, \tilde{C}$ the images of S, Q , and C under this contraction.

The calculation of $\pi_1(\mathbb{P}^2 \setminus (C \cup S \cup Q))$ will be realized by using the projection from the point $O := P \cap Q$, where P is the tangent to C at its cusp as usual. The real picture is as in Figure (16), except that we should take the line $S = \overline{pq} = \{z = 0\}$ into consideration. Now, the only points of intersection of the line S with C are the points p and q . Hence, the only singular lines of the projection from the point O are P, Q , and R , where R is the simple tangent line from O to C , as shown in the previous section.

Choose a fiber F and a base B as in Figure (16)-I, and put $F' := F \setminus (C \cup S \cup \{O\})$, $B' := B \setminus (P \cup Q \cup R)$. The situation is illustrated in Figure (18). Choose the generators $a_1, a_2, \dots, a_{k+1}, b, \gamma$ for $\pi_1(F')$, and the generators α, β for $\pi_1(B')$ as in Figure (19).

The monodromy around the singular fiber Q gives

$$\beta^{-1} a_i \beta = \begin{cases} a_{i+1}, & 1 \leq i \leq k, \\ (\delta \gamma) a_1 (\delta \gamma)^{-1}, & i = k+1, \end{cases} \quad [\beta, b] = 1,$$

where $\delta := a_{k+1} a_k \cdots a_1$. From the monodromy around R we get,

$$a_{k+1} = b.$$

These relations yields a presentation of the group $\pi_1(\mathbb{P}^2 \setminus (C \cup S \cup Q))$.

Applying Fujita's Lemma to the elementary transformations applied at the points r_i , we obtain the relation $\beta b^n = 1$. Imposing this relation on the relations found above, we find that $a_1 = a_2 = \cdots = a_{k+1} = b$, and $[b, \gamma] = 1$. We conclude that the group is abelian.

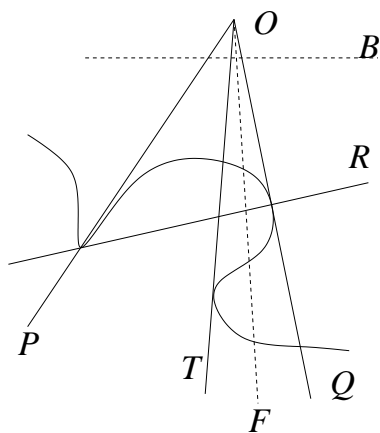


Figure 18

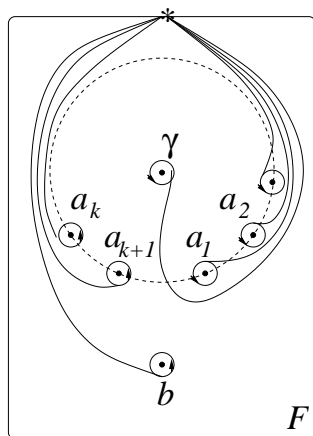


Figure 19

3.7 Groups of the curves (7) in Theorem 2.9

Let C be the curve $xy^{k-1} - z^k = 0$, with $p := [1 : 0 : 0]$ as its cusp, $q := [0 : 1 : 0]$ as its inflection point, and P, Q as the tangents to C at these points. Blow-up the point $O := P \cap Q$, to get the Hirzebruch surface X_1 , with the horizontal section E_1 . Perform an elementary transformation at q_1 , followed by an elementary transformation at $E_2 \cap P_2$. Now for $i = 3, 4, \dots, n+2$ perform elementary transformations applied at the points $E_i \cap P_i$, followed by elementary transformations applied at some points $s_i \in Q_i \setminus E_i$ for $i = n+3, n+4, \dots, 2n+2$. We end up with a Hirzebruch surface X_{2n+3} with $E_{2n+3} = -1$. Contraction of E_{2n+3} gives the projective plane \mathbb{P}^2 .

We shall use the projection from the point O to find the fundamental group. This projection has, in addition to P and Q , a third singular fiber R , which is a simple tangent from O to C at a unique point. The setting is same as in Case 4, see Figure (14). Choose a base fiber close to Q , and take the generators for the base and the fiber as in Figure (15). The monodromy around Q gives the relations

$$\beta^{-1}a_i\beta = \begin{cases} a_{i+1}, & 1 \leq i \leq k-1, \\ \delta a_1 \delta^{-1}, & i = k, \end{cases}$$

where $\delta := a_k a_{k-1} \cdots a_1$.

Now, without finding the monodromy around R or P , notice that an obvious application of Fujita's lemma gives $a_1\beta$ as a meridian of Q_1 . Subsequent elementary transformations on Q_i are applied at some points $s_i \in Q_i \setminus E_i$, so that $a_1\beta$ stays to be a meridian of Q_i . Imposing the relation $a_1\beta = 1$ on the above relations gives $\beta^{-1} = a_1 = a_2 = \cdots = a_k$. But, the group $\pi_1(\mathbb{P}^2 \setminus (C \cup P \cup Q))$ is generated by the elements $a_1, a_2, \dots, a_k, \beta$. Hence, the fundamental groups of the curves of the family (7) are abelian.

3.8 Groups of the curves (8) in Theorem 2.9

Let C be the curve $xy^2 - z^3 = 1$. Pick an arbitrary smooth point $q \in C$ which is not the inflection point of C , and let Q be the tangent of C at q . Put $P := \overline{pq}$, where p is the cusp of C .

By Bezout's theorem, the line Q intersect C at a second point, say r .

Blowing-up the point q , we get the Hirzebruch surface X_1 , with the horizontal section E_1 . For $i = 1, 2, \dots, n-1$, perform elementary transformations at the points r_i , followed by elementary transformations performed at the points $E_i \cap P_i$ for $i = n, n+1, \dots, 2n-2$. We end up with the Hirzebruch surface X_{2n-2} with $E_{2n-1}^2 = -1$. Contraction of E_{2n+1} gives the projective plane \mathbb{P}^2 back. Denote as usual by $\tilde{P}, \tilde{Q}, \tilde{C}$ the images of P, Q, C under this contraction. Then \tilde{C} is a curve of the family (8).

Let us show that the group $\pi_1(\mathbb{P}^2 \setminus (C \cup P \cup Q))$ is abelian, thereby showing that the groups of the curves of the family (8) are abelian.

Consider the *singular* projection from the cusp p . A generic fiber of this projection intersects $C \cup P \cup Q$ at two points, i.e. $\pi_1(\mathbb{P}^2 \setminus (C \cup P \cup Q))$ is generated by two elements. These two points meet at the point r , which is

a transversal intersection of Q with C . This implies that the corresponding generators commute. It follows that $\pi_1(\mathbb{P}^2 \setminus (C \cup P \cup Q))$ is abelian. \square

3.9 Groups of the curves in Theorem 2.10

Let C be the curve $(yz - x^2)^2 - x^3y = 0$.

Lemma 4. (*Fenske [5]*) C is a rational cuspidal quartic with cusps at the point $p := [0 : 0 : 1]$ of type $[2_2]$, and at the point $r := [0 : 1 : 0]$ of type $[2_1]$. This curve has an inflection point of order 3 at the point $q := [-576 : -4096 : 135]$.

Let P be the tangent to C at the cusp p , and let Q be the inflectional tangent line at q . The cusp p is the only intersection point of P with C , whereas Q intersect C at a second further point, say s . It is clear that the point $O := P \cup Q$ does not lie on C . Blowing-up O , we get the Hirzebruch surface X_1 , with a section E_1 such that $E_1^2 = -1$. Now for $i = 1, 2, \dots, n$ perform elementary transformations at the points s_i , followed by elementary transformations applied at the points $E_i \cup P_i$ for $i = n + 1, n + 2, \dots, 2n + 1$. We end up with the Hirzebruch surface X_{2n+1} with $E_{2n+1}^2 = -1$. Contraction of $E_{2n+1}^2 = -1$ gives the projective plane \mathbb{P}^2 . Denote by \tilde{C} , \tilde{P} , \tilde{Q} the images of C_{2n+1} , P_{2n+1} , Q_{2n+1} under this contraction. Then \tilde{C} is the desired curve of degree $3n + 4$.

To find the fundamental group of $C \cup P \cup Q$, we shall use the projection from the center O . Let R be the line \overline{Or} . Then, clearly P , Q and R are singular fibers of this projection. That there are no other singular fibers can be seen by looking at the dual picture. By the class formula, the degree of the curve C^* dual to C is 4. The line O^* intersects C^* at its simple cusp Q^* with multiplicity 2. Since the line P intersects C with multiplicity 4 at the cusp p , O^* intersects C^* with multiplicity 2 at the cusp P^* of C^* . (That P^* is a cusp of C^* of multiplicity 2 can be seen by using the parameterization $[t^2 : t^4 : 1 + t]$ of C .) By Bezout's theorem, this accounts for all the intersection points of the line O^* with C^* .

To get a better picture of the curve, we apply the transformation $[x : y : z] \rightarrow [x : y : y + z]$, and then pass to the affine coordinate system in $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$, where $L_\infty = \{z = 0\}$. In these coordinates, C is parameterized as $(t^2/(1 + t + t^4), t^4/(1 + t + t^4))$, the cusp p is the point $(0, 0)$, the cusp r is the point $(0, 1)$, and the flex q is the point $(576/3961, 4096/3961)$. It turns out that the point s , the second point of intersection of Q with C , is real. The configuration $C \cup P \cup Q$ is pictured in Figure (20). Pick F , B as in Figure (20), put $F' := F \setminus (C \cup \{O\})$, $B' := B \setminus (P \cup Q \cup R)$, and choose the generators a, b, c, d of F' and the generators α, β, γ of B' as in Figure (21).

The monodromy around Q gives the relations

$$\beta^{-1}b\beta = c, \quad \beta^{-1}c\beta = d, \quad \beta^{-1}d\beta = dcb(dc)^{-1},$$

and the relation

$$\beta^{-1}a\beta = a \Leftrightarrow [\beta, a] = 1.$$

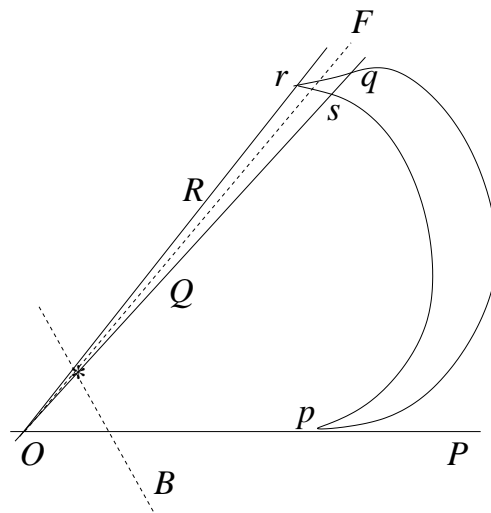


Figure 20

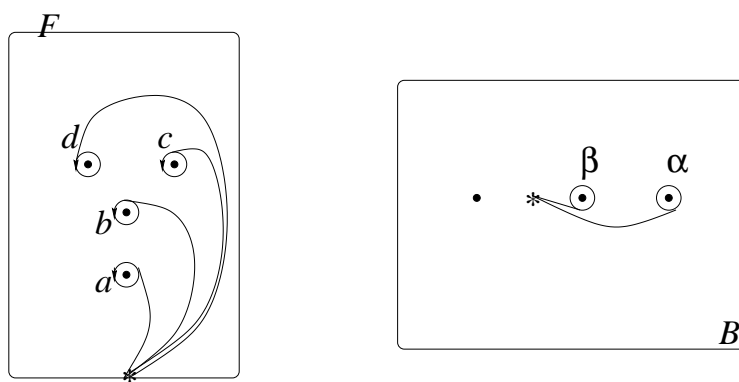


Figure 21

Using these relations, one can express the generators c, d in terms of b and β , and one easily deduces the relation

$$(b\beta)^3 = (\beta b)^3.$$

Finally, the monodromy around P gives the cusp relation

$$aba = bab.$$

This completes the presentation of $\pi_1(\mathbb{P}^2 \setminus (C \cup P \cup Q))$. Note that the loop α can be expressed by using the relation

$$dcba\alpha\beta = 1 = \beta^{-2}(b\beta)^3a\alpha,$$

which implies that $[\alpha, \beta] = 1$, this latter can also be derived by an application of Fujita's lemma at the point O .

By Fujita's lemma, $a^n\beta$ is a meridian of \tilde{Q} , and $\alpha^{n+1}\beta^n$ is a meridian of \tilde{P} . Imposing the corresponding relations on the above presentation we get

$$\begin{aligned} \pi_1(\mathbb{P}^2 \setminus \tilde{C}) = \langle a, b, \alpha, \beta \mid & (b\beta)^3 = (\beta b)^3 \quad (\mathcal{R}_1), \quad aba = bab \quad (\mathcal{R}_2), \\ & \beta a^n = 1 \quad (\mathcal{R}_3), \quad \alpha^{n+1}\beta^n = 1 \quad (\mathcal{R}_4), \quad \beta^{-2}(b\beta)^3a\alpha = 1 \quad (\mathcal{R}_5) \rangle \end{aligned}$$

Note that the relation $[\beta, a] = 1$ follows from the relation (\mathcal{R}_3) and therefore is not written in the above presentation.

In order to simplify this presentation, put $b\beta =: \gamma$. Then the relation (\mathcal{R}_1) can be expressed as $[\beta, \gamma^3] = 1$, or, substituting $\beta = a^{-n}$, as $[a^n, \gamma^3] = 1$. On the other hand, $b = \gamma\beta^{-1} = \gamma a^n$, so that the cusp relation (\mathcal{R}_2) becomes

$$a\gamma a^n a = \gamma a^n a \gamma a^n \Leftrightarrow a\gamma a = \gamma a^{n+1} \gamma.$$

Using (\mathcal{R}_5) , we can express α as follows:

$$\alpha = a^{-1}\gamma^{-3}a^{-2n}.$$

Substituting this in (\mathcal{R}_4) we get, using $[a^n, \gamma^3] = 1$,

$$(a^{-1}\gamma^{-3}a^{-2n})^{n+1}a^{n^2} = 1 \Leftrightarrow a^{3n^2+2n}(\gamma^3a)^{n+1} = 1.$$

So we have obtained the presentation

$$\pi_1(\mathbb{P}^2 \setminus \tilde{C}) = \langle a, \gamma \mid a\gamma a = \gamma a^{n+1} \gamma, \quad [a^n, \gamma^3] = a^{3n^2+2n}(\gamma^3a)^{n+1} = 1 \rangle.$$

Put $G := \pi_1(\mathbb{P}^2 \setminus \tilde{C})$. Then one has

$$G(n) = G(a^n) = \langle a, \gamma \mid a\gamma a = \gamma a \gamma, \quad (\gamma^3 a)^n = 1 \rangle.$$

Applying the transformation $x := a\gamma$, $y := a\gamma a$, and imposing the relation $x^3 = 1$ we get a surjection $G \twoheadrightarrow T$, where

$$T := \langle x, y \mid y^2 = x^3 = x^{2(n+1)} = (xy^{-1})^n = 1 \rangle,$$

so that if $3|(n+1)$, then one has a surjection $G(n) \twoheadrightarrow T_{2,3,n}$ onto the triangle group, which implies that $G(n)$ is big for $n \geq 7$.

A more economical way of writing the last relation in the presentation of G is as follows: one has $a^{n+1} = \gamma^{-1}a\gamma a\gamma^{-1}$. Hence,

$$\begin{aligned} a^{3n^2+2n}(\gamma^3 a)^{n+1} &= a^{2n^2+n-1}a(\gamma^3 a^{n+1})^3 \\ &= (a^{n+1})^{2n-1}a\gamma^2(\gamma a^{n+1}\gamma \cdot \gamma \cdot \gamma a^{n+1} \dots \gamma a^{n+1}\gamma)\gamma^{-1} \\ &= (\gamma^{-1}a\gamma a\gamma^{-1})^{2n-1}a\gamma^2(a\gamma a)^{n+1}\gamma^{-1}, \end{aligned}$$

so that one can replace the last relation by the relation

$$(\gamma^{-1}a\gamma a\gamma^{-1})^{2n-1}a\gamma(\gamma a)^{2n+2}\gamma^{-1}.$$

Remark. By the following lemma, the group G is actually a quotient of the braid group \mathbb{B}_3 on three strands.

Lemma 5. *For any $n \in \mathbb{Z}$, one has $\langle a, b \mid aba = ba^{n+1}b \rangle \simeq \mathbb{B}_3$.*

Proof. Applying the transformation $(a, b) \rightarrow (x := a, y := ba^n)$, with inverse $(x, y) \rightarrow (a = x, b = yx^{-n})$, the above relation becomes

$$x y x^{-n} x = y x^{-n} x^{n+1} y x^{-n},$$

which is nothing but the braid relation $xyx = yxy$. \square

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